Cosmology from General Relativity

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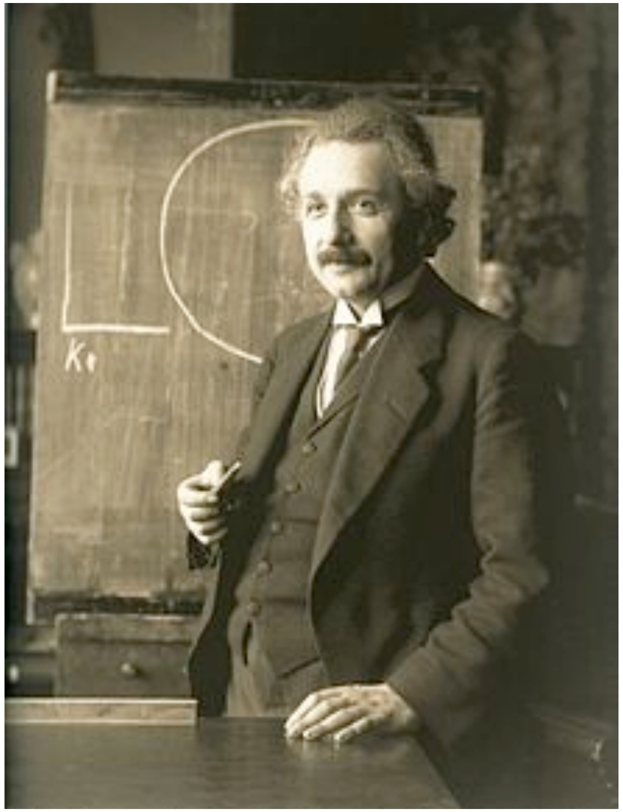
Oxford, Sept 2015

“An important contribution of the general theory of relativity to cosmology has been to keep out theologians by a straightforward application of tensor analysis.”

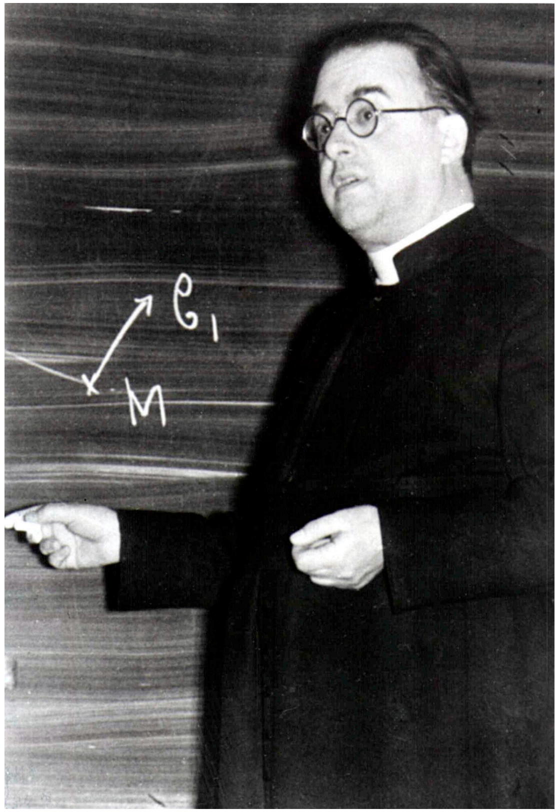
E. Schucking

Albert Einstein

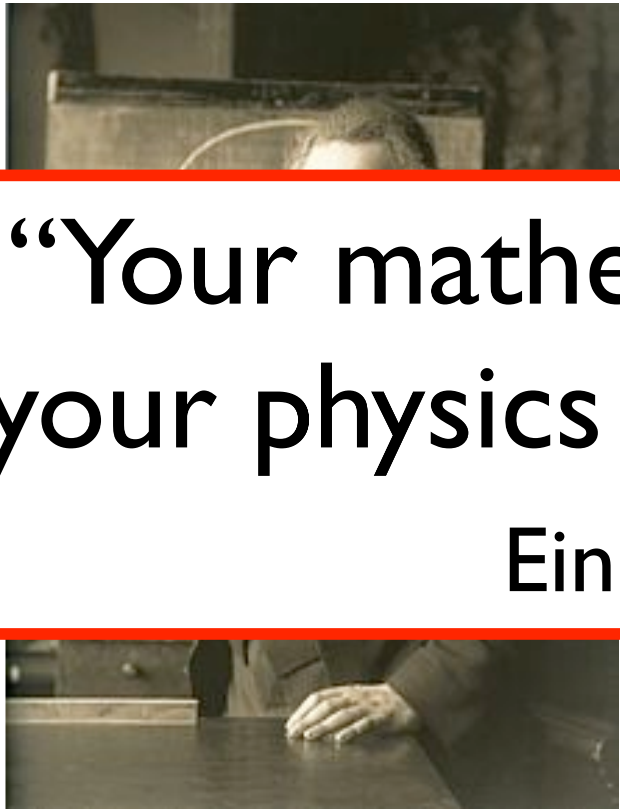


http://www.bhm.ch/de/news_04a.cfm?bid=4&jahr=2006

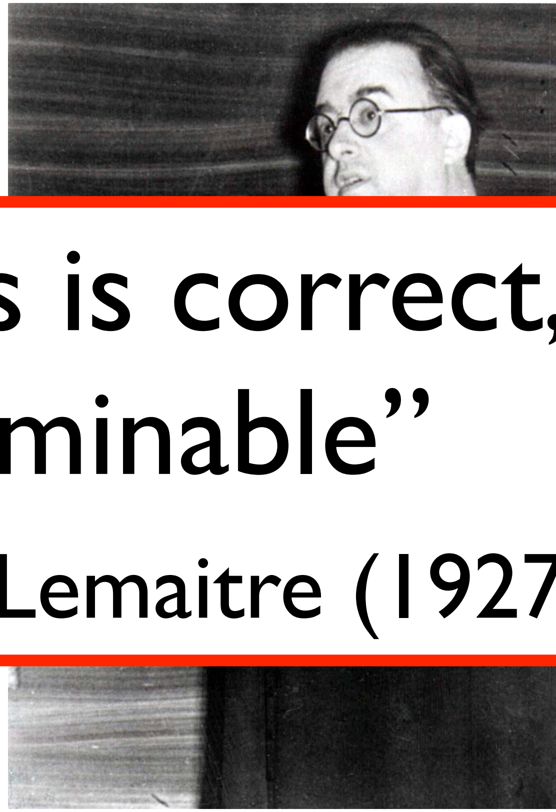
Georges Lemaitre



Albert Einstein



Georges Lemaitre



"Your mathematics is correct, your physics is abominable"

Einstein to Lemaitre (1927)

http://www.bhm.ch/de/news_04a.cfm?bid=4&jahr=2006

Newtonian Gravity

$$\vec{F} = -m\vec{\nabla}\Phi$$

and $$\nabla^2\Phi = 4\pi G\rho$$

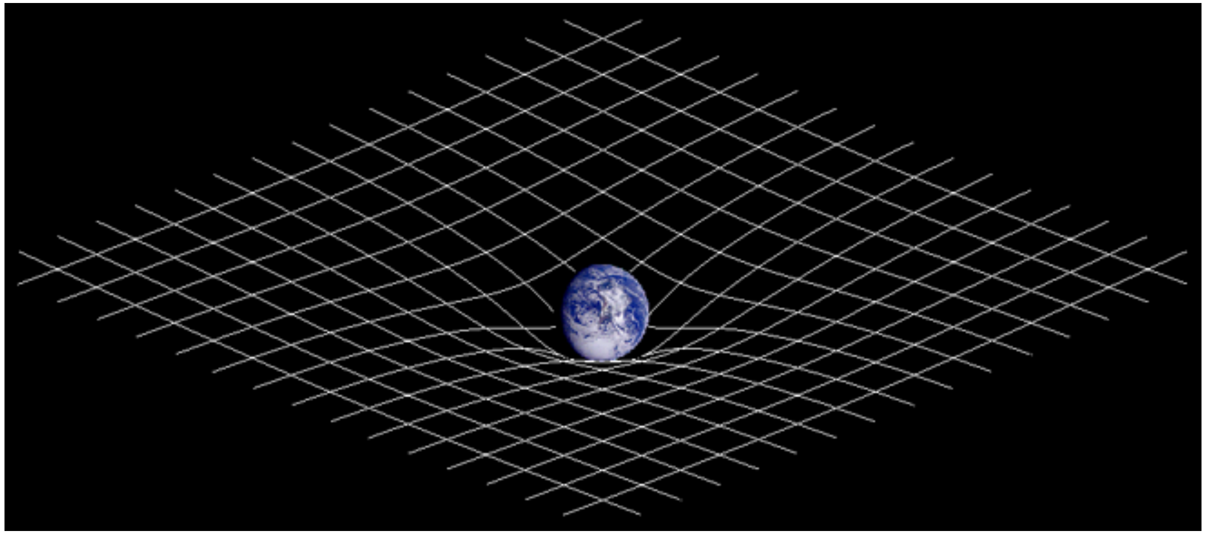
$$m\frac{d^2\vec{r}_1}{dt^2} = \vec{F}$$

fix

Given ρ solve for Φ, calculate $\vec{F}$ and then solve for $\vec{r}_1$.

General Relativity

"Geometry"= "Energy"



http://en.wikipedia.org/wiki/Spacetime

"Space-time tell matter how to move;
matter tells space-time how to curve"

John Archibald Wheeler

General Relativity in practice

4 by 4 symmetric matrix of functions of space-time:

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

$g_{\alpha\beta}$ → The metric of space time

General Relativity in practice

The Ricci tensor can be determined from the metric but ...

$$
\begin{aligned}
R_{00} \;=\; & \frac{1}{2}\frac{d}{dx^0}\left(g^{00}\frac{d}{dx^0}g_{00}\right) - \frac{d}{dx^0}\left(g^{01}\frac{d}{dx^1}g_{00}\right) \\
& + \frac{d}{dx^0}\left(g^{01}\frac{d}{dx^0}g_{01}\right) + g^{00}\frac{d}{dx^0}\left(g^{00}\frac{d}{dx^0}g_{00}\right) \\
& + g^{01}\frac{d}{dx^0}\left(g^{21}\frac{d}{dx^2}g_{00}\right) + \text{another} \;\; 283 \;\; \text{terms...}
\end{aligned}
$$

where $g^{\alpha\beta} \;=\; [g_{\mu\nu}]^{-1}$

General Relativity in practice

"Geometry" "Energy"

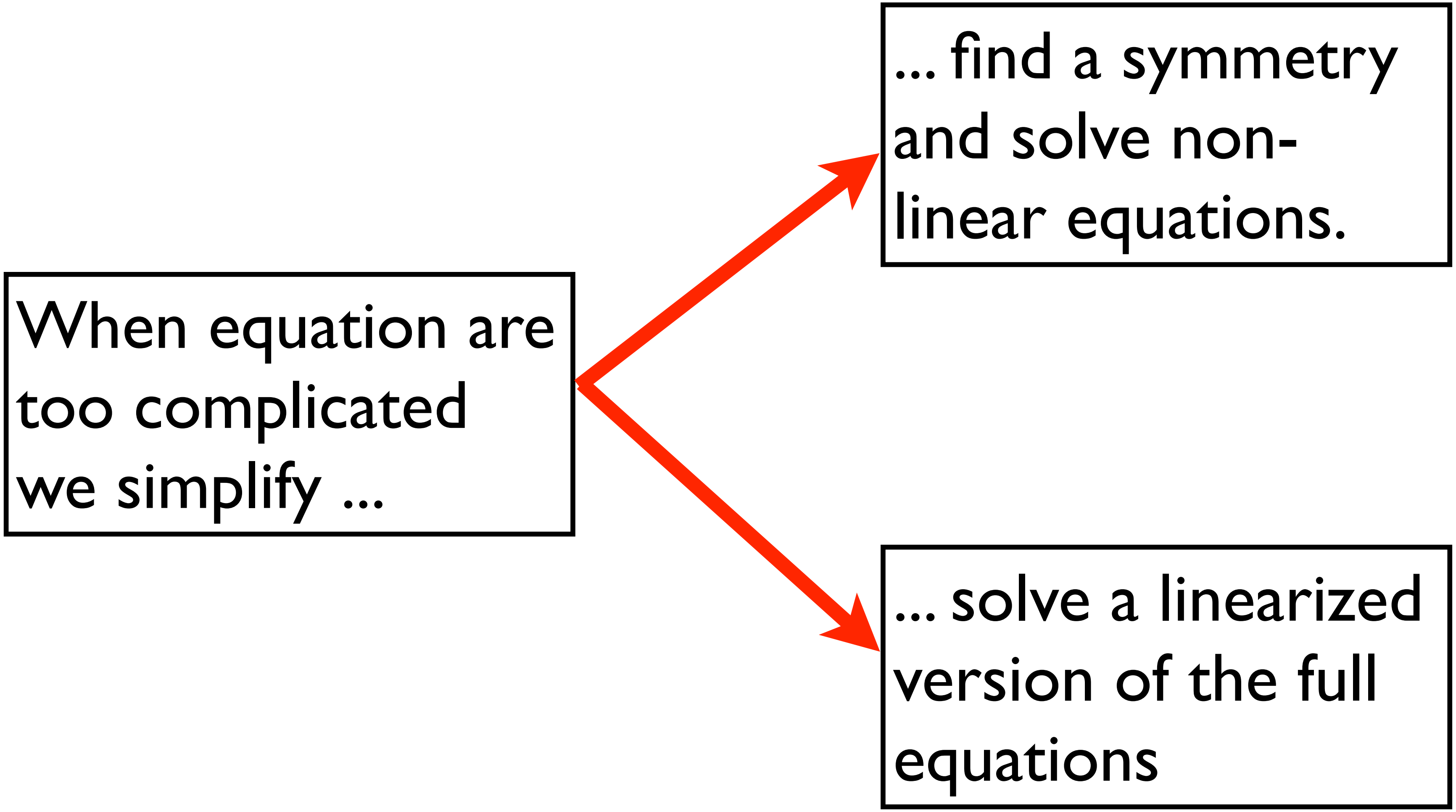
$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

and the energy momentum tensor:

$$T_{00} = \rho c^2 \qquad T_{11} = P g_{11}$$

mass density pressure

Cosmology


When equation are too complicated we simplify ...
... find a symmetry and solve non-linear equations.
... solve a linearized version of the full equations

Cosmology

Homogeneity- at any given time, the Universe looks exactly the same at any point in space.

Cosmological Principle:

Isotropy- at any given time, and at any point, the Universe looks the same in any direction.

Cosmology

Energy is "smoothly" distributed: $\rho(t)$ and $P(t)$.

10 functions of space-time ...

...1 function of time

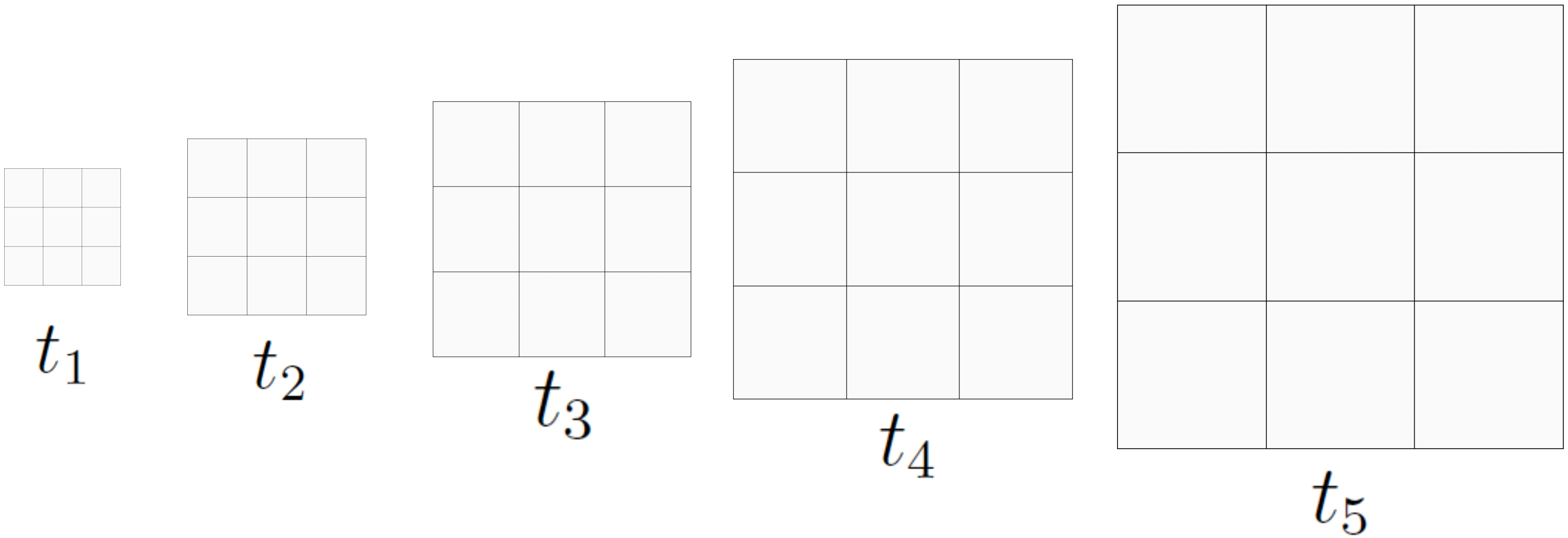
$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}$$

scale factor

Cosmology

Homogeneous and isotropic metric "rescales" space over time.

$$ds^2 \;=\; -c^2dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$



Cosmology

Dramatic simplification of the equations ...

curvature tensor

$$R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}$$

$$R_{11} = R_{22} = R_{33} = \frac{1}{c^2}(a\ddot{a} + 2\dot{a}^2)$$

Einstein equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$$

$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3\frac{P}{c^2})$$

Cosmology

Gas of massive particles:

energy density today

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3a}$$

energy

$$\rho = \frac{E}{V} = \frac{Mc^2}{a^3L^3} \propto \frac{1}{a^3}$$

volume

Solution: $a(t) \propto t^{2/3}$

Universe is dynamic *not* static.

$a = 0$ at $t = 0$ → Initial singularity.

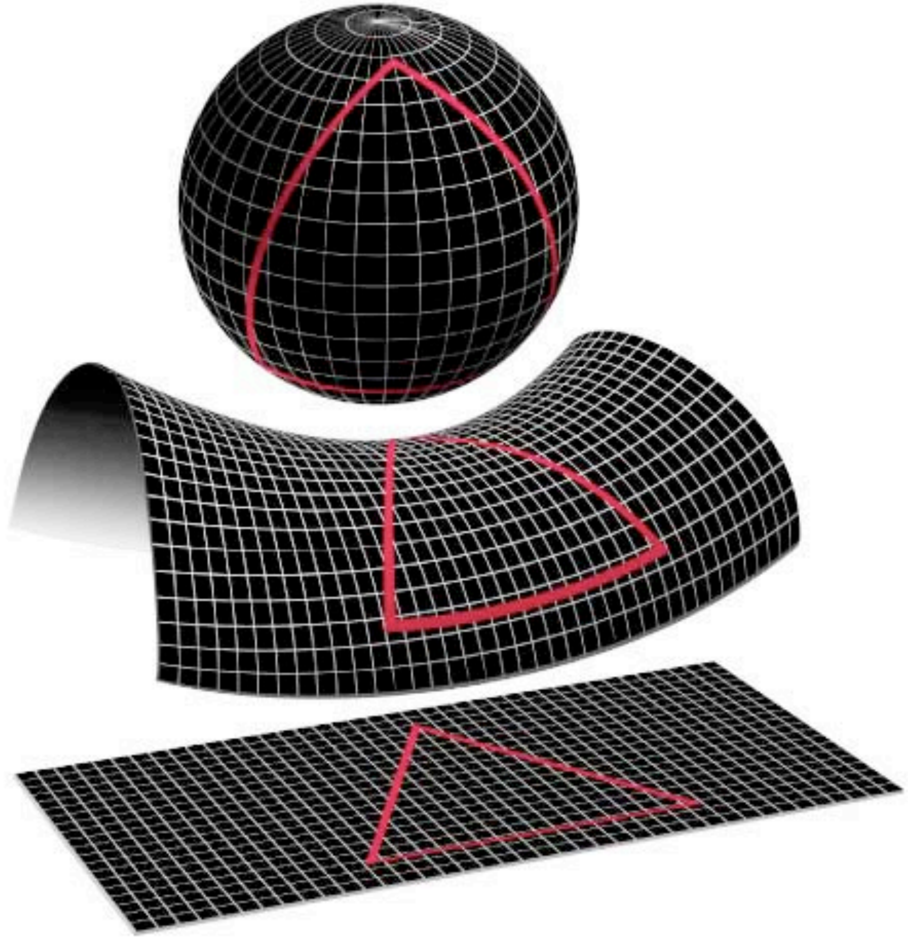
Cosmology

Three homogeneous and isotropic spaces:

Hyper-spherical (positive curvature)

Hyperbolic (negative curvature)

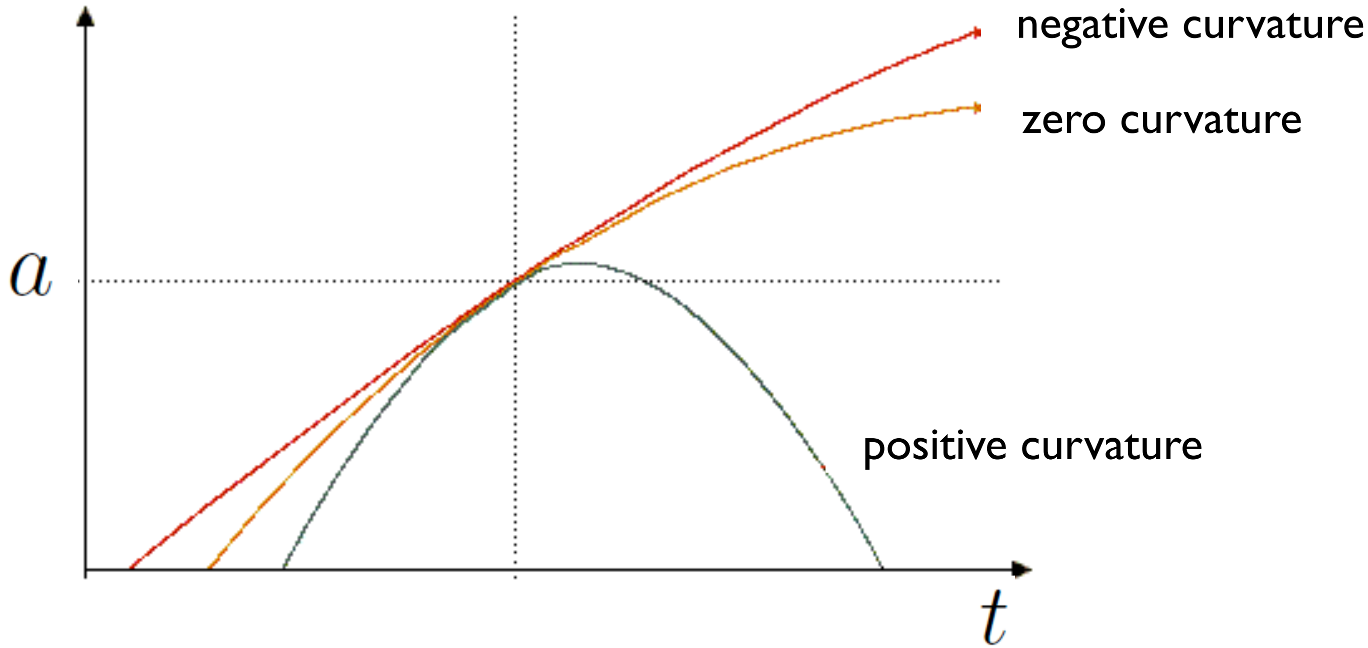
Euclidean or "flat" (no curvature)



Cosmology

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

$$|k| \propto \frac{1}{(\text{"Radius of curvature"})^2}$$



Cosmology

A ball in a gravitational potential:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_0}{3a^3} - \frac{kc^2}{a^2} \qquad \dot{a}^2 = \frac{8\pi G\rho_0}{3a} - kc^2$$

Mass in volume ($L = 1$): $M = \rho_0 \times V = \rho_0 \times \frac{4\pi}{3}L^3$

$$\frac{1}{2}\dot{a}^2 - \frac{GM}{a} = -\frac{kc^2}{2} = E_{\rm tot}$$

kinetic energy

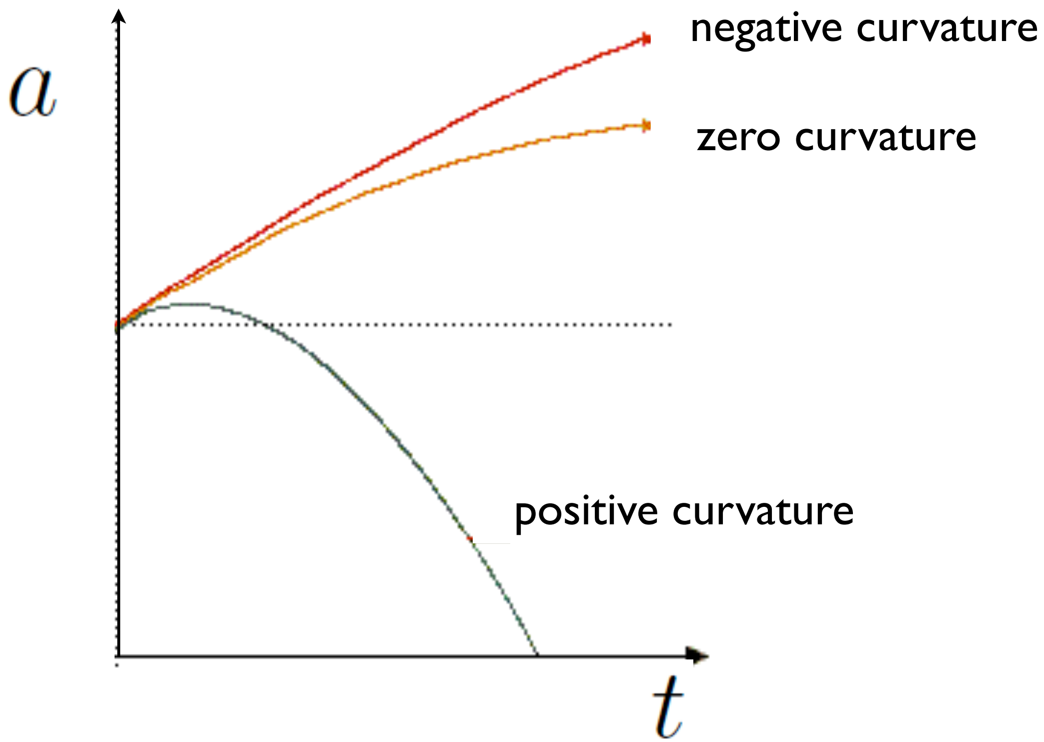
potential energy

Sign of k determines if orbit is bound or unbound.

Cosmology

A ball in a gravitational potential:

$$\frac{1}{2}\dot{a}^2 - \frac{GM}{a} = -\frac{kc^2}{2} \quad = \quad E_{\mathrm{tot}}$$



Cosmology

Can the expansion accelerate?

Λ cosmological constant

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta} - \Lambda g^{\alpha\beta}$$

New terms in Einstein equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

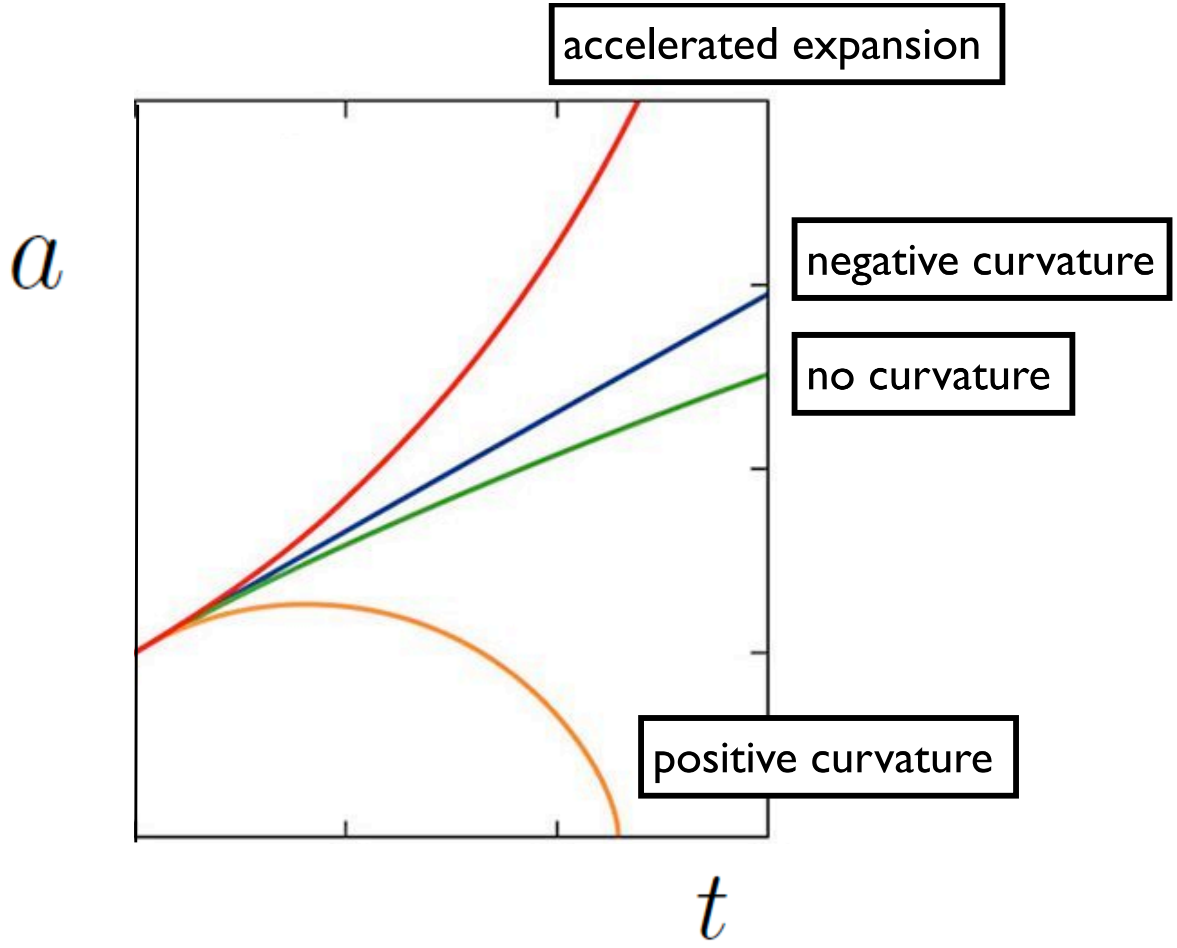
Positive acceleration term

accelerated expansion

$$\frac{\ddot{a}}{a} = -\frac{4\pi G \rho_0}{a^3} + \frac{\Lambda}{3} \rightarrow a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)$$

Cosmology

Ball in a potential: $$\frac{1}{2}\dot{a}^2 - \frac{GM}{a} - \frac{\Lambda}{6}a^2 = E_{\rm tot}$$



Cosmology

Einstein's static universe:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3} \quad \text{and} \quad \frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{a^3} + \frac{2}{3}\Lambda$$

solve: $\rho = \dfrac{\Lambda c^2}{4\pi G}$ and $a = \sqrt{\dfrac{k}{\Lambda}}$

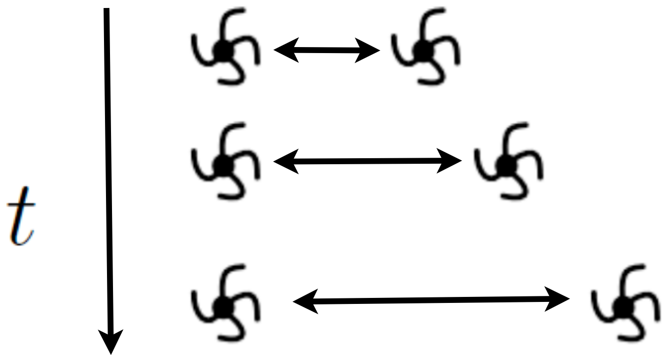
but ... unstable!

Cosmology

Universe must evolve!

Cosmology

How do we observe the evolution?

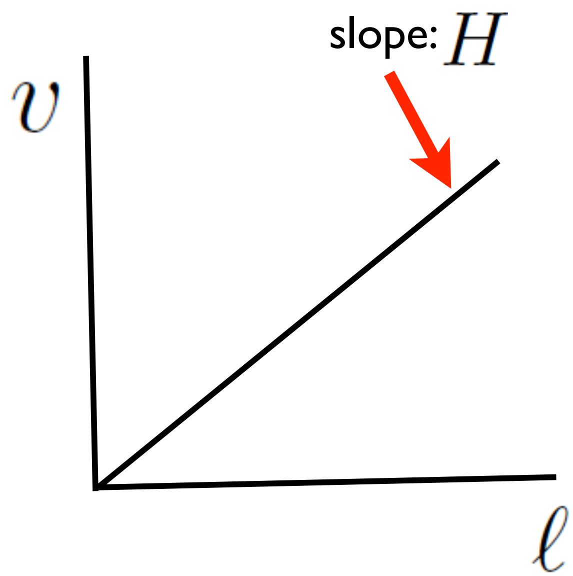


$$ds^2 = -c^2dt^2 + a^2dx^2$$

$$\ell = a\Delta x$$

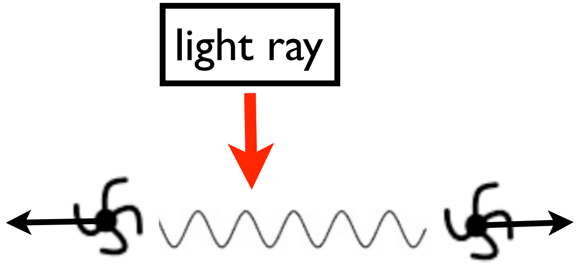
$$v = \dot{\ell} = \frac{d}{dt}(a\Delta x) = \dot{a}\Delta x = \frac{\dot{a}}{a}\ell$$

$$v = H\ell$$



Cosmology

The redshift effect:



Doppler effect

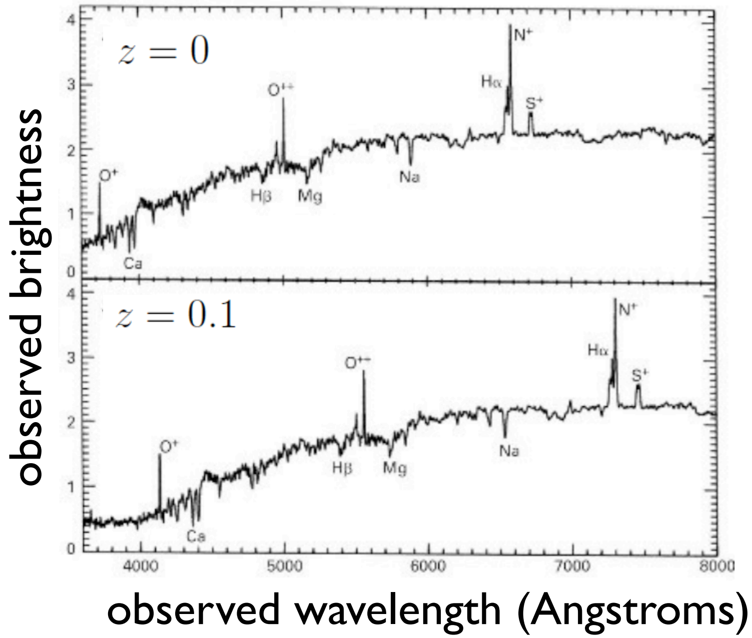
$$\frac{d\lambda}{\lambda} = \frac{dv}{c} = \frac{\dot{a}}{a}\frac{dr}{c} = \frac{\dot{a}}{a}dt = \frac{da}{a}$$

$$\lambda \propto a$$

$$\frac{\lambda_{\rm ob}}{\lambda_{\rm em}} = 1 + z$$

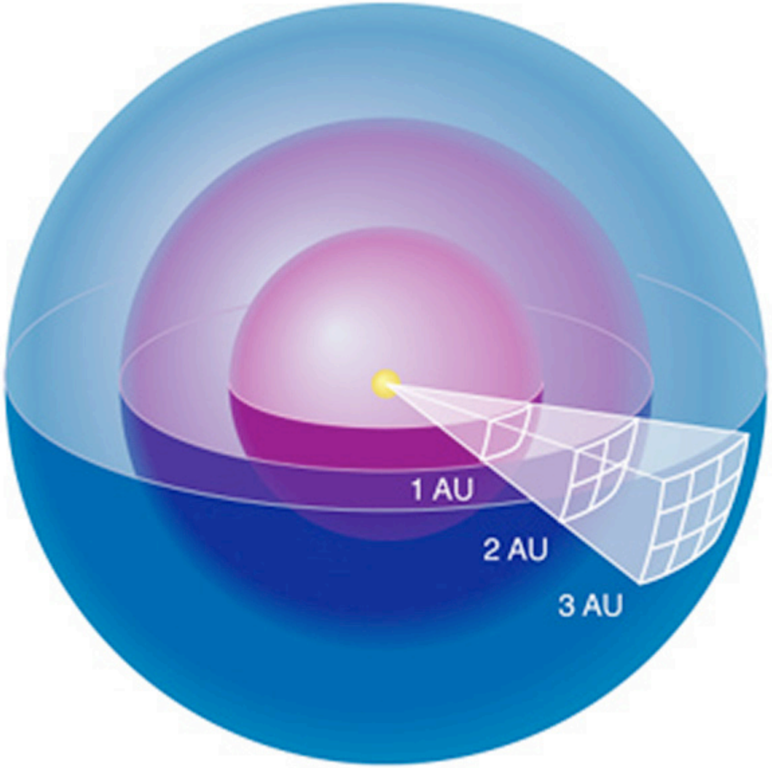
$$\frac{a_0}{a} = 1 + z$$

Redshift



Cosmology

The further away an object the dimmer it is.



luminosity

$$L = \frac{dE}{dt}$$

flux

$$F = \frac{L}{A}$$

area

$$A \propto (D_L)^2$$

luminosity distance

Expanding Universe

NEBULÆ.

BY V. M. SLIPHER, PH.D.

(Read April 13, 1917.)

In addition to the planets and comets of our solar system and the countless stars of our stellar system there appear on the sky many cloud-like masses—the nebulæ. These for a long time have been generally regarded as presenting an early stage in the evolution of the stars and of our solar system, and they have been carefully studied and something like 10,000 of them catalogued.

TABLE I.

RADIAL VELOCITIES OF TWENTY-FIVE SPIRAL NEBULÆ.

Nebula.	Vel.	Nebula.	Vel.
N.G.C. 221	− 300 km.	N.G.C. 4526	+ 580 km.
224	− 300	4565	+1100
598	− 260	4594	+1100
1023	+ 300	4649	+1090
1068	+1100	4736	+ 290
2683	+ 400	4826	+ 150
3031	− 30	5005	+ 900
3115	+ 600	5055	+ 450
3370	+ 780	5194	+ 270
3521	+ 730	5236	+ 500
3623	+ 800	5866	+ 650
3627	+ 650	7331	+ 500
4258	+ 500		

Vesto Slipher 1917

21 out of 25 nebula were redshifted ...

Expanding Universe

Lemaitre (1927)

Utilisant les 42 nébuleuses figurant dans les listes de Hubble et de Strömberg ([1]), et tenant compte de la vitesse propre du soleil (300 Km. dans la direction $\alpha = 315°$, $\delta = 62°$), on trouve une distance moyenne de 0,95 millions de parsecs et une vitesse radiale de 600 Km./sec, soit 625 Km./sec à 10^6 parsecs ([2]).

Nous adopterons donc

$$\frac{R'}{R} = \frac{v}{rc} = \frac{625 \times 10^5}{10^6 \times 3{,}08 \times 10^{18} \times 3 \times 10^{10}} = 0{,}68 \times 10^{-27}\ \mathrm{cm}^{-1} \qquad (24)$$

Eddington's translation of Lemaitre (1931)

Therefore

$$\frac{v}{c} = \frac{\delta t_2}{\delta t_1} - 1 = \frac{R_2}{R_1} - 1 \quad . \quad . \quad . \quad . \quad (22)$$

is the apparent Doppler effect due to the variation of the radius of the universe. *It equals the ratio of the radii of the universe at the instants of observation and emission, diminished by unity.*

v is that velocity of the observer which would produce the same effect. When the light source is near enough, we have the approximate formulæ

$$\frac{v}{c} = \frac{R_2 - R_1}{R_1} = \frac{dR}{R} = \frac{R'}{R}dt = \frac{R'}{R}r$$

where r is the distance of the source. We have therefore

$$\frac{R'}{R} = \frac{v}{cr} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From a discussion of available data, we adopt

$$\frac{R'}{R} = 0{\cdot}68 \times 10^{-27}\ \mathrm{cm.}^{-1} \quad . \quad . \quad . \quad (24)$$

Expanding Universe

Hubble (1929)

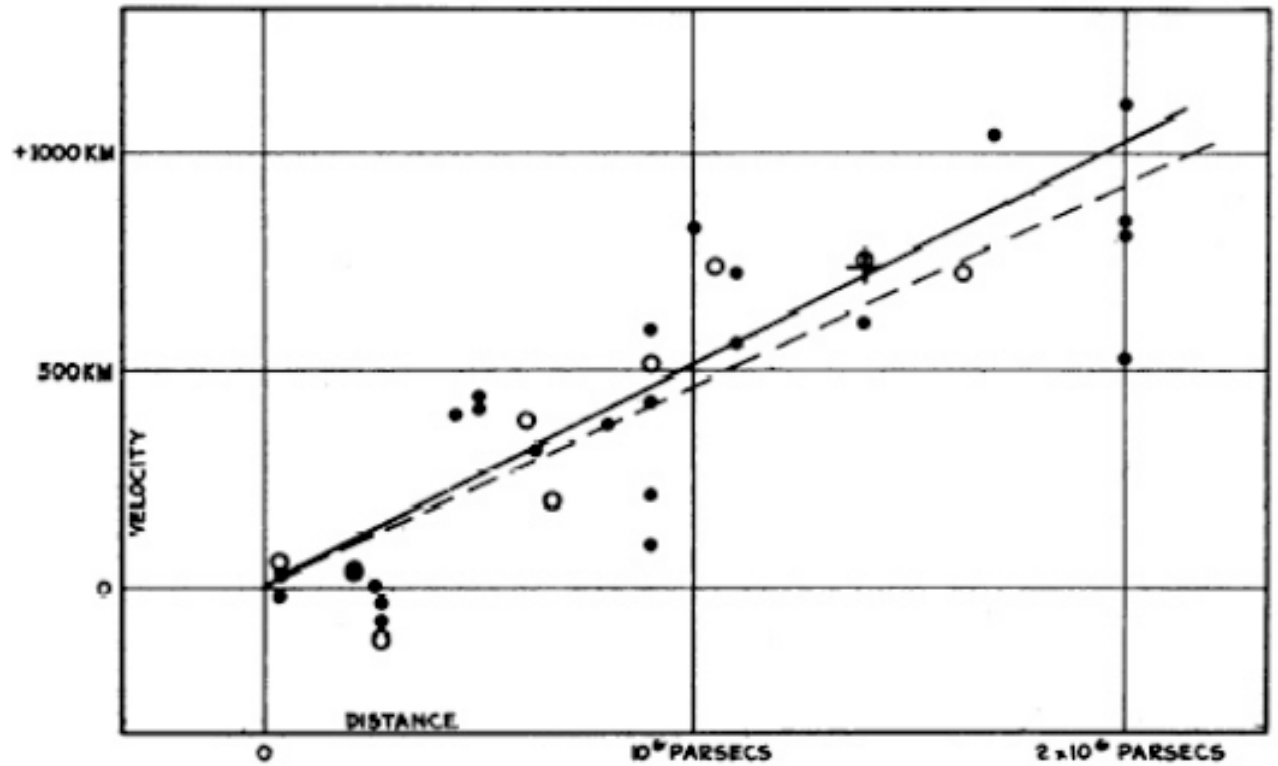
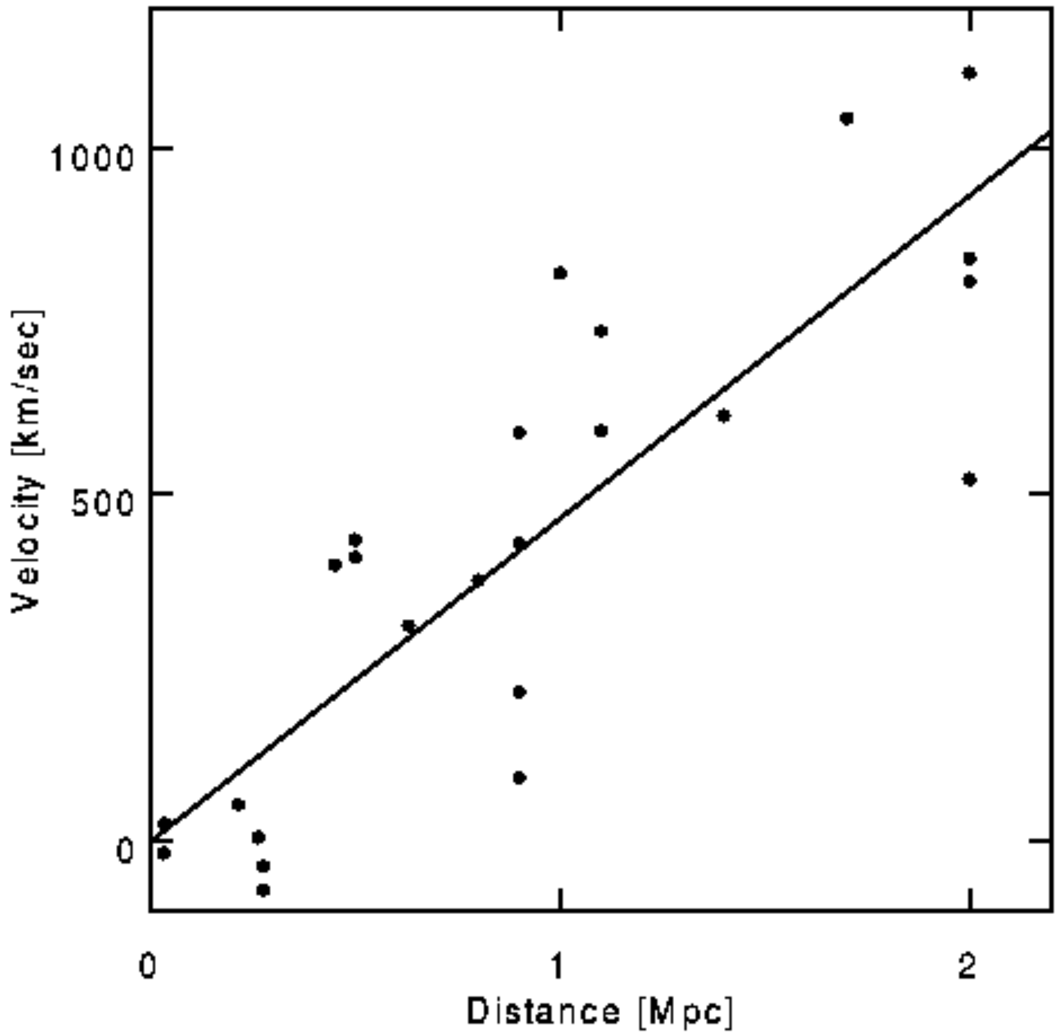


FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.

1 parsec = 3.26 light years

Expanding Universe

Hubble (1929)



Hubble constant: $H_0 = 550\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$

Expanding Universe

The age problem.

Redshift effect: $v = H\ell$ where $H = \dfrac{\dot{a}}{a}$

Solution to Einstein equation: $a \propto t^{2/3}$ → $\dot{a} \propto \dfrac{2}{3} t^{-1/3}$

Invert $H_0 = \dfrac{2}{3}\dfrac{1}{t_0}$ to find age of the universe $t_0 = \dfrac{2}{3}\dfrac{1}{H_0}$

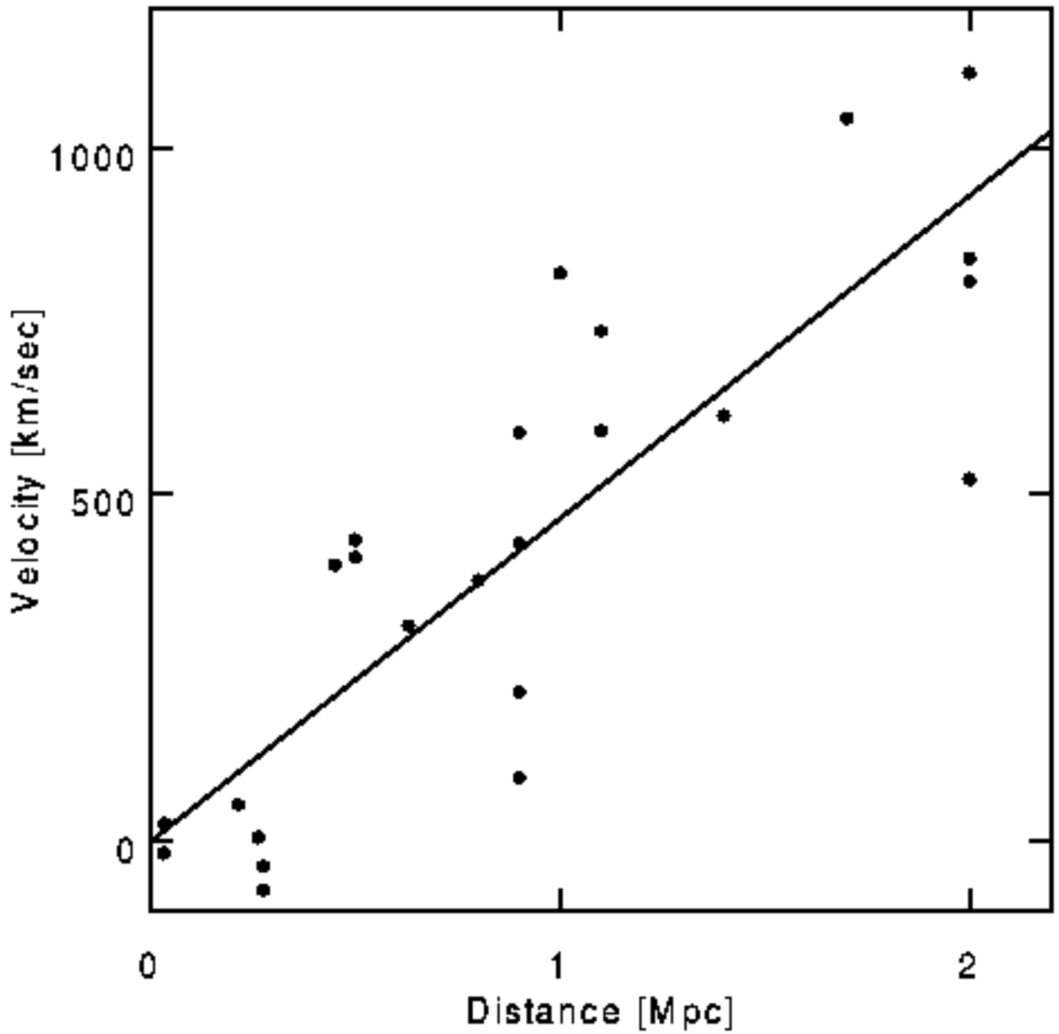
Hubble's measurement: $H_0 = 550\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$

Age: $t_0 \sim 1$ Gigayear

Universe is too young!

Expanding Universe

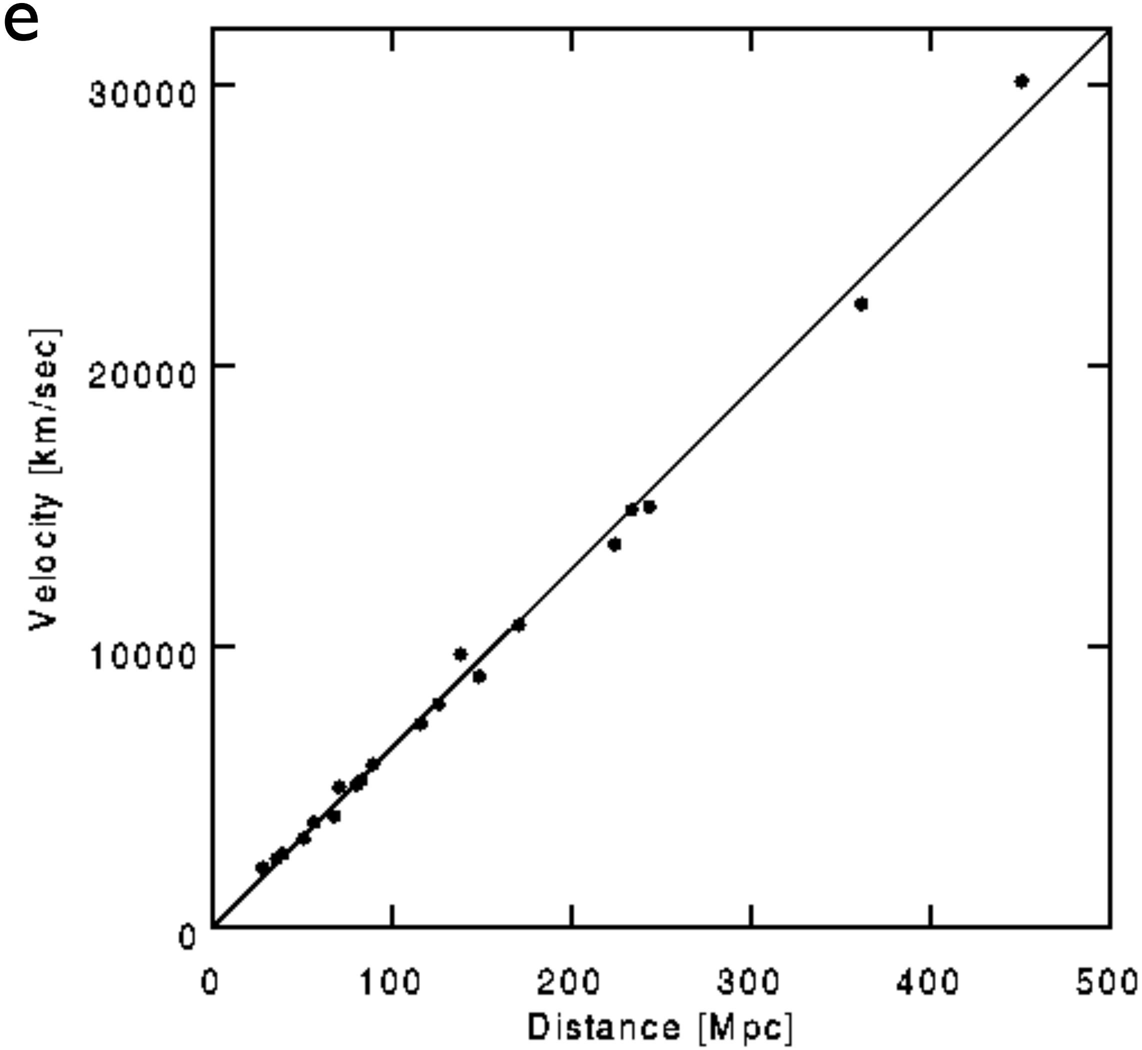
Hubble (1929)



Hubble constant: $H_0 = 550\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$

Expanding Universe

Reiss et al (1995)



Hubble constant: $H_0 = 80\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$

A Hot Big Bang

Adiabatic expansion: $PV^{\gamma} = \text{constant}$

pressure → P; volume → V; adiabatic index → γ

$$\gamma = \frac{c_p}{c_V}$$

Ideal gas law: $PV = nRT$

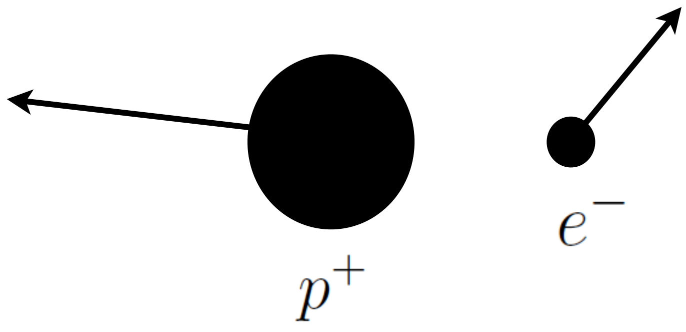
$$TV^{\gamma-1} = \text{constant}$$

A gas of photons: $\gamma = \frac{4}{3}$ → $T \propto \frac{1}{a}$

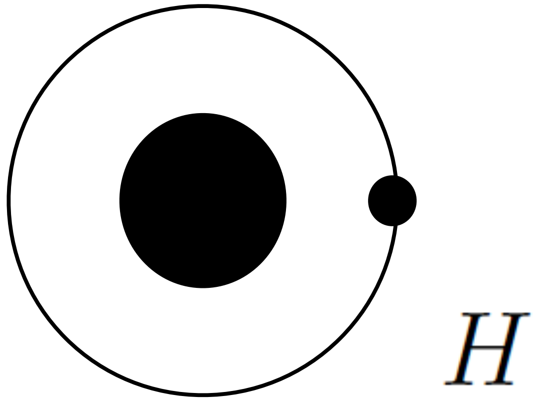
The universe was hotter in the past

A Hot Big Bang

High Temperature



Low Temperature



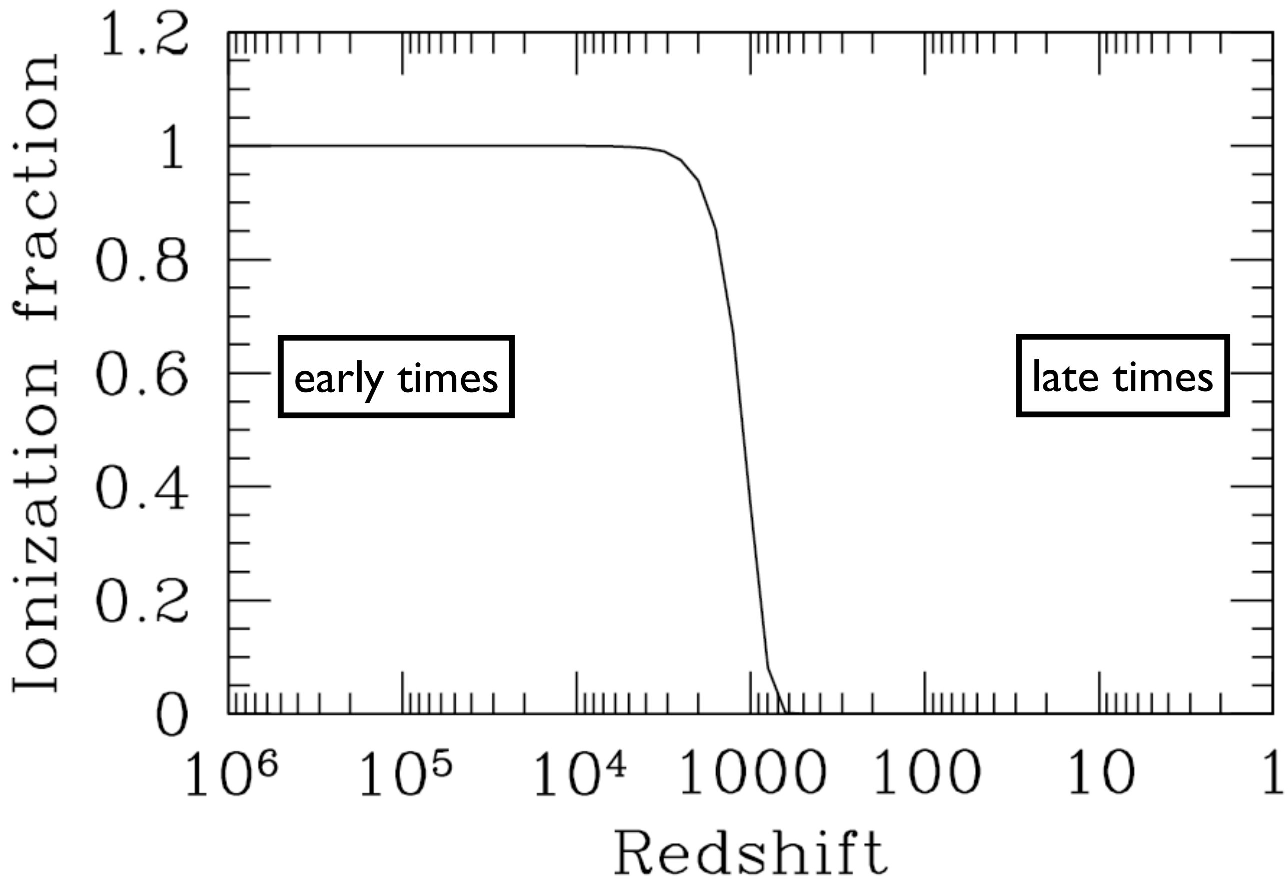
Binding energy: $E = 13.6\,\mathrm{eV} = 2.2 \times 10^{-18}\mathrm{Joules}$

A Hot Big Bang

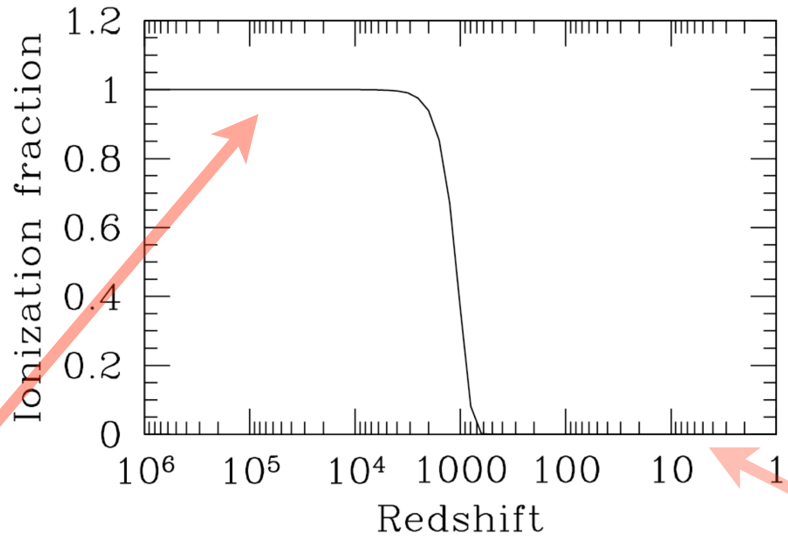
number density of protons

$$X = \frac{n_p}{n_p + n_H}$$

number density of Hydrogen

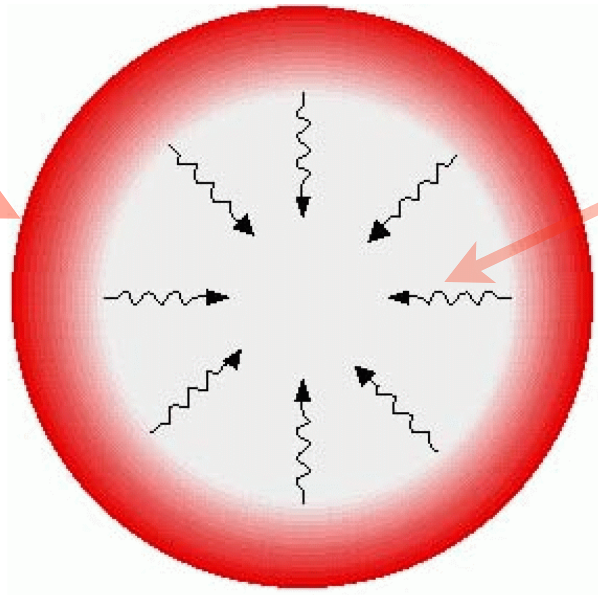


A Hot Big Bang

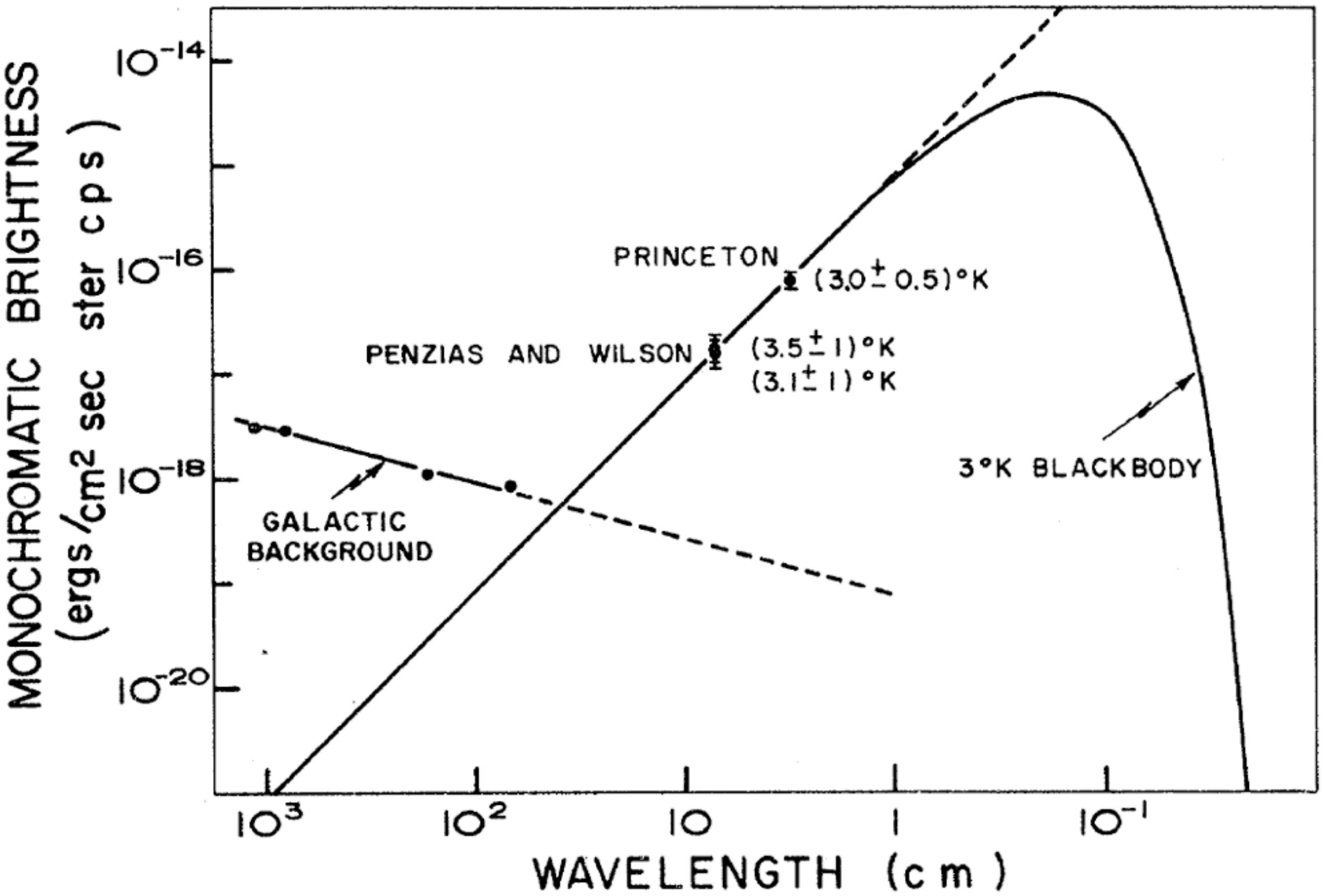


Tight coupling:

Free streaming:

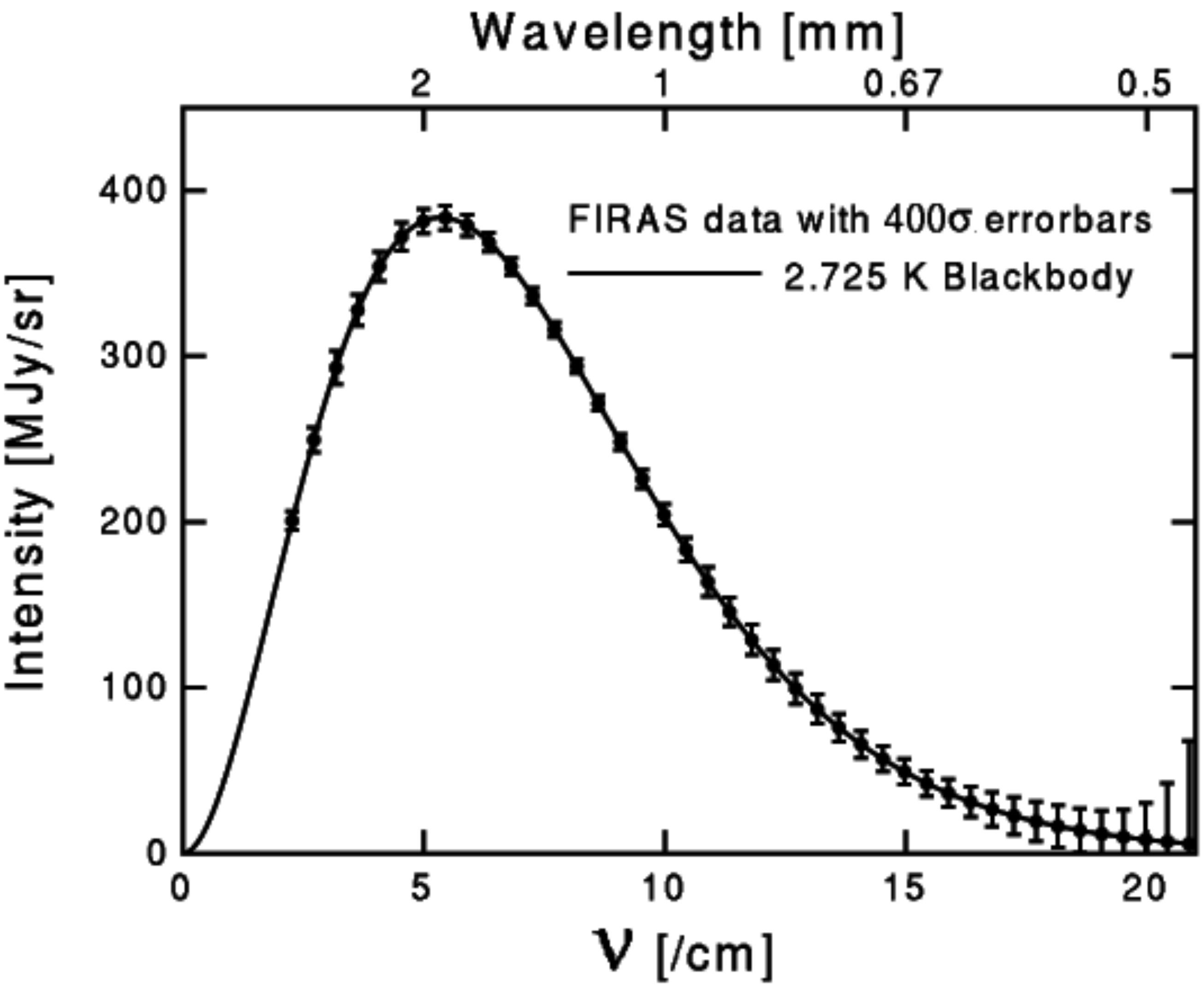


A Hot Big Bang



P. Roll & D. Wilkinson (1966)

A Hot Big Bang



An accelerating Universe

Taylor expand the scale factor:

$$a(t) = a(t_0) + \dot{a}(t_0)[t - t_0] + \frac{1}{2}\ddot{a}(t_0)[t - t_0]^2 + \cdots$$

Use $D_L \simeq c(t_0 - t)$, $\dfrac{a(t)}{a(t_0)} = (1+z)^{-1}$ and $H_0 = \dfrac{\dot{a}(t_0)}{a(t_0)}$ (Hubble constant), $q_0 = -\dfrac{\ddot{a}(t_0)a(t_0)}{\dot{a}^2(t_0)}$ (deceleration rate)

To get:

$$\frac{1}{1+z} = 1 - H_0\frac{D_L}{c} - \frac{q_0 H_0^2}{2}\left(\frac{D_L}{c}\right)^2 + \cdots$$

An accelerating Universe

Taylor expand the scale factor:

$$z = H_0 \frac{D_L}{c} + H_0^2 \left(\frac{q_0}{2} + 1 \right) \left(\frac{D_L}{c} \right)^2 + \cdots$$

or

$$\frac{D_L}{c} = \frac{1}{H_0} \left[z - \left(1 + \frac{q_0}{2} \right) z^2 + \cdots \right]$$

Quadratic corrections to Hubble's law tell us about acceleration!

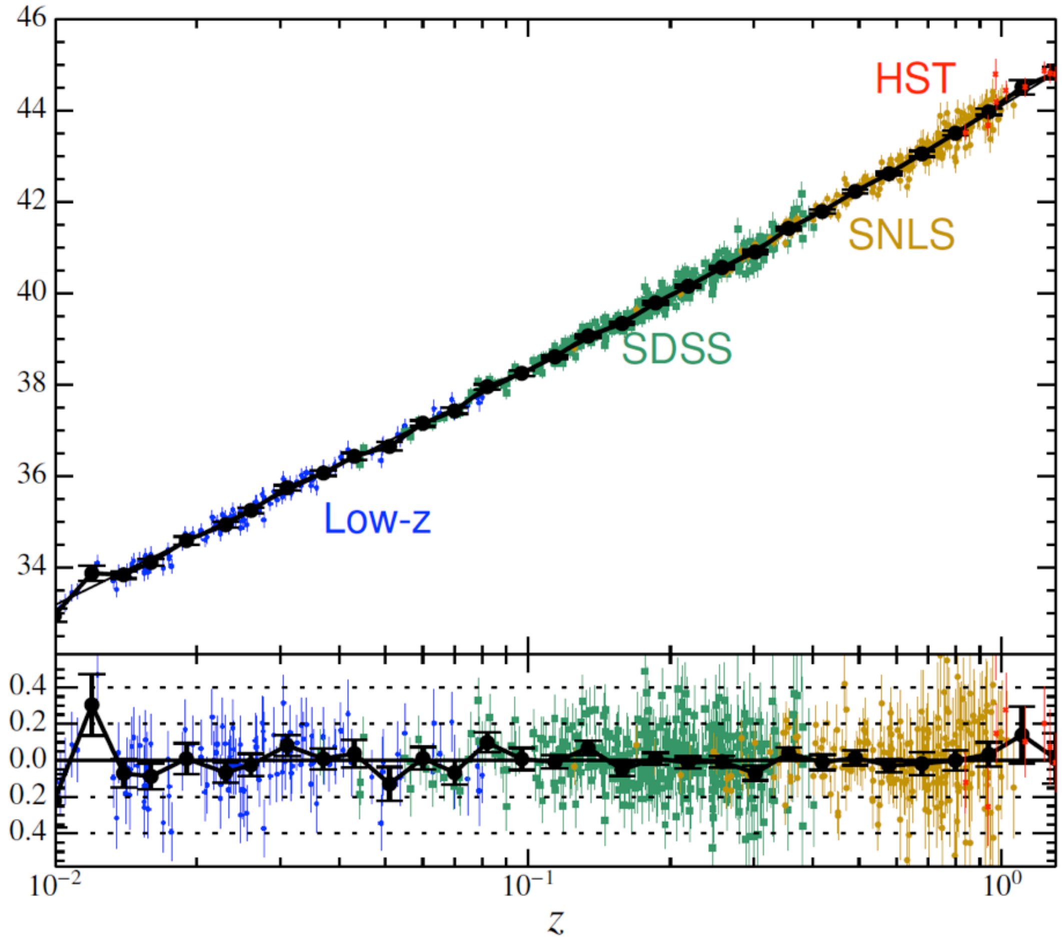
An accelerating Universe

Supernovae IA

apparent magnitude

$$\mathcal{M} \propto -\log D_L$$

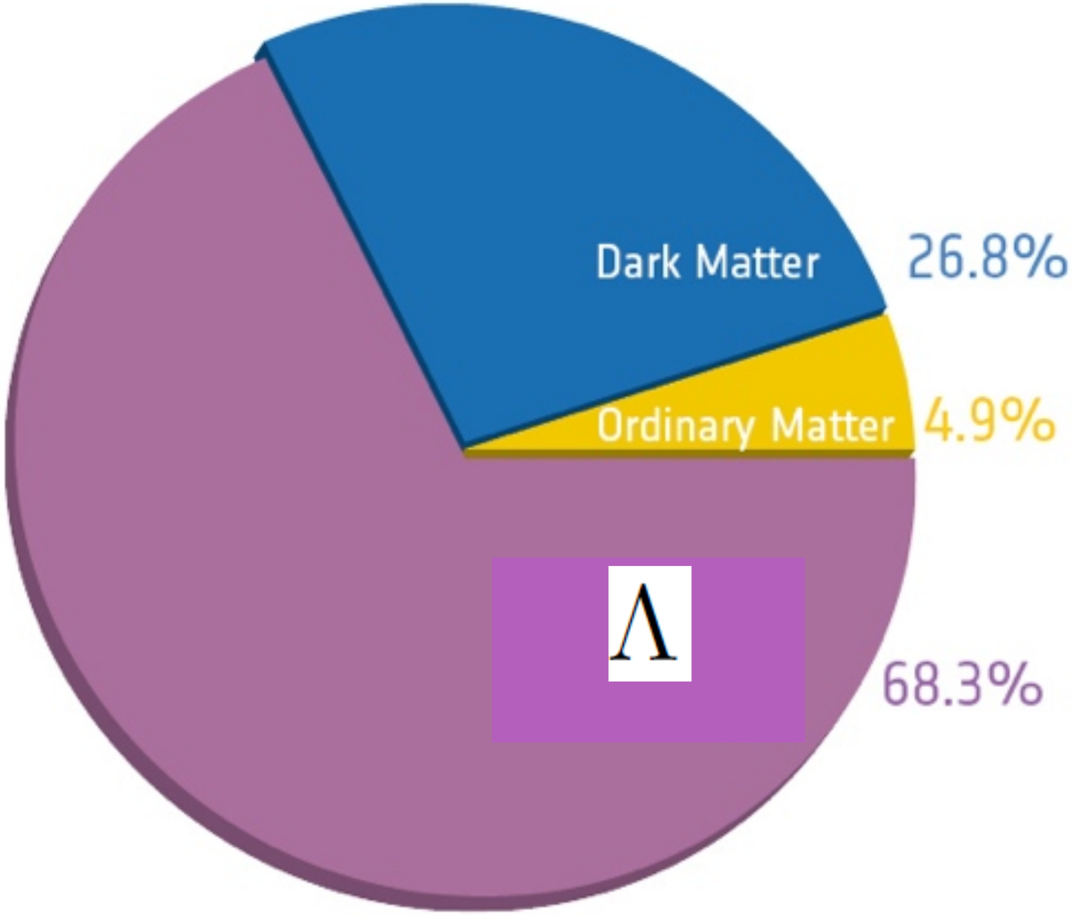
luminosity distance



$$1 + z = \frac{1}{a}$$

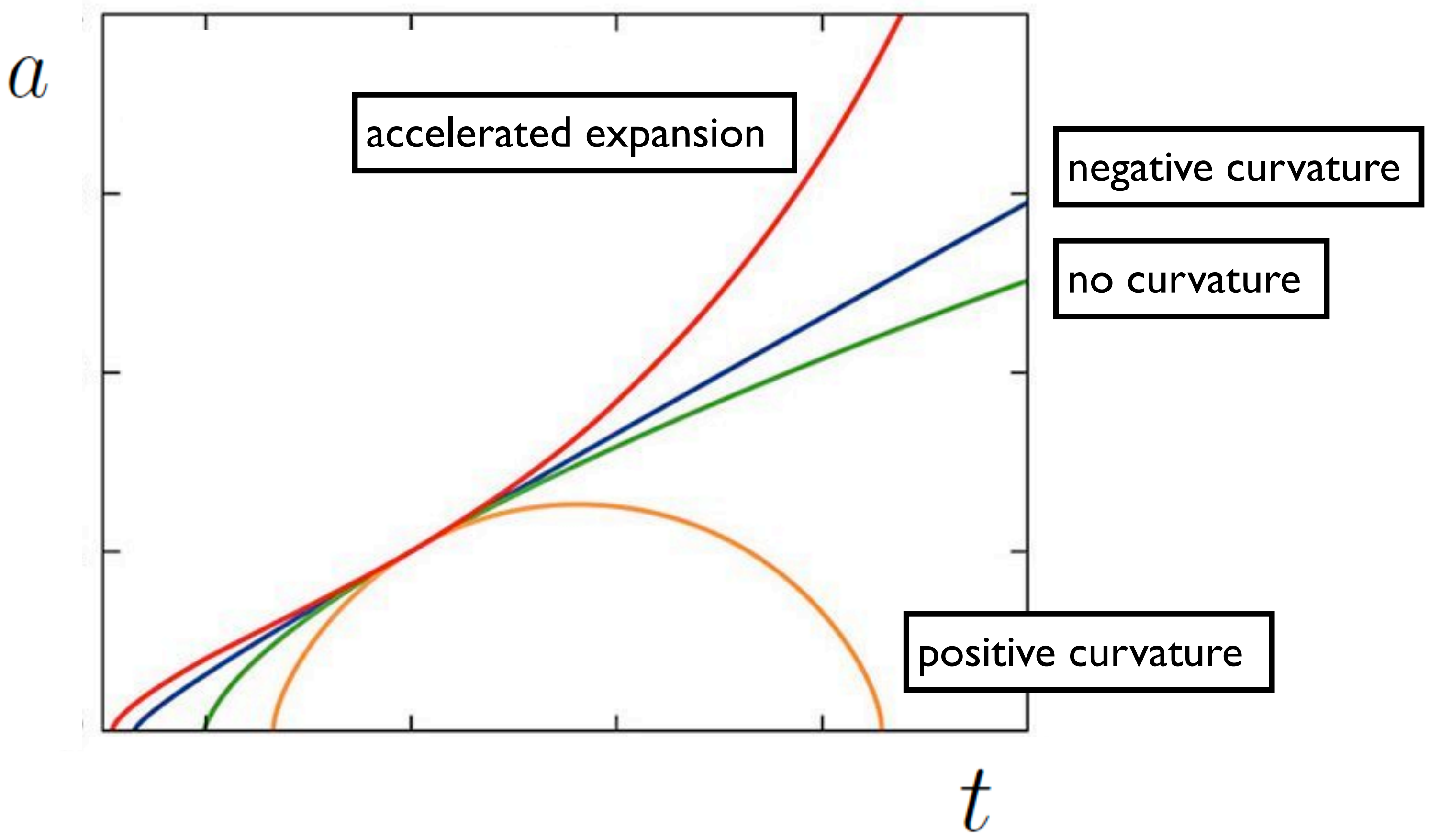
Betoule et al (2014)

An accelerating Universe



An accelerating Universe

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{a^3} + \frac{\Lambda}{3} \qquad a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)$$



An accelerating Universe

Small Cosmological Constant is "unnatural"

Quantum vacuum fluctuations give a cosmological constant:

$$\Lambda \to \Lambda + \Lambda_c \quad \text{with} \quad \Lambda_c \sim G \int_0^M \omega(k) \frac{d^3k}{h^3} \sim \frac{GM^4c}{h^3}$$

Electron: $m_e = 0.5\mathrm{MeV}$

$$\Lambda_C = 4.5 \times 10^{-22}\,\mathrm{m}^{-2} \gg \Lambda = 10^{-52}\,\mathrm{m}^{-2}$$

Too much accelerated expansion!

An accelerating Universe

Alternatives to Λ

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

modify gravity

$$G_{\alpha\beta} + F[g_{\mu\nu}, R_{\mu\nu}, \cdots]$$

corrections to the Einstein's equations

new forces

gravitational collapse

dark energy

ϕ — scalar field (like the Higgs)

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

potential energy

Λ

$$\dot{\phi} \sim 0$$

slow roll

"Remember that the most beautiful things in the world are the most useless."

John Ruskin