GENERAL RELATIVITY

WHAT IT IS AND WHY EINSTEIN CONCEIVED IT THUS

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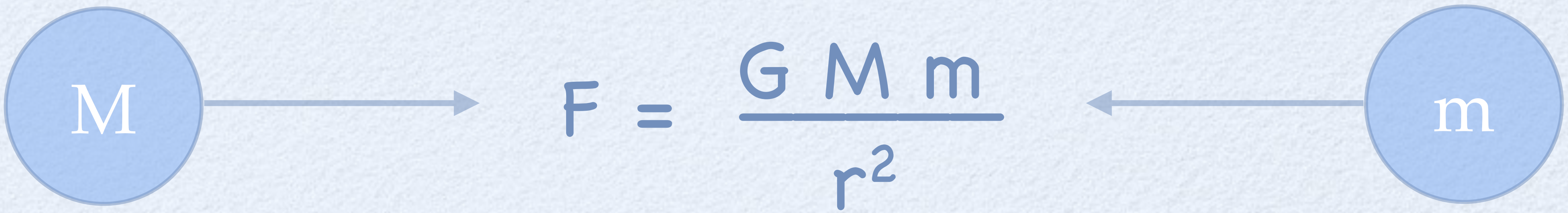
General Relativity

1. The picture in circa 1890
2. Special Relativity & Observers
3. Equivalence & Mach
4. Metrics & Parallel Transport
5. Geodesics & Curvature
6. Einstein's Equations

1.1 The picture in circa 1890

Newtonian Gravity is a *force law*



$$F = \frac{G\,M\,m}{r^2}$$

Space and time form a fixed canvas upon which processes happen. The action of F is `at a distance' and instantaneous.

In 1687 this was a triumph — suddenly a whole swathe of observations are explained by a single principle.

In 1890 it still works pretty well but there are a couple of issues…

1.2 The picture in circa 1890

Le Verrier used the Newtonian law to analyse in great detail the orbit of Uranus and concluded

- The orbit is indeed inconsistent with the then known objects in the Solar System
- The anomaly could be removed by hypothesising a new planet — Neptune, whose exact location he announced 31 August 1846 & was confirmed by Galle and d'Arrest on 23 September

But the trick failed with the perihelion of Mercury problem, the necessary correcting planet was not present!

1.3 The picture in circa 1890

Electromagnetism was a triumph of 19th century physics, culminating in Maxwell's unification of 1861/2

$$\nabla . B = 0 \qquad \nabla . E = 4\pi\rho$$

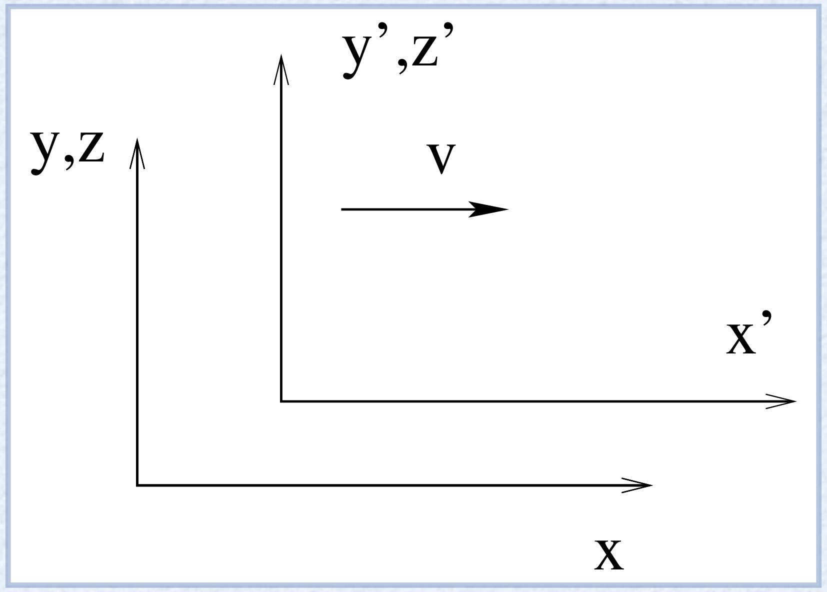
$$\nabla \times B = \frac{\partial E}{\partial t} + 4\pi J \qquad \nabla \times E = - \frac{\partial B}{\partial t}$$

- Light is an electromagnetic wave travelling with a finite speed - reassuring as Fizeau had measured it in 1849
- Signals travel at the speed of light and we don't have 'action at a distance' - feels right
- But Maxwell's equations transform differently from Newton's laws between relatively moving frames

2.1 Special Relativity and Observers

Lorentz and Lamor realised that to fix this time must transform non-trivially…

~~$x' = x - vt$~~
~~$t' = t$~~
~~$y' = y$~~
~~$z' = z$~~



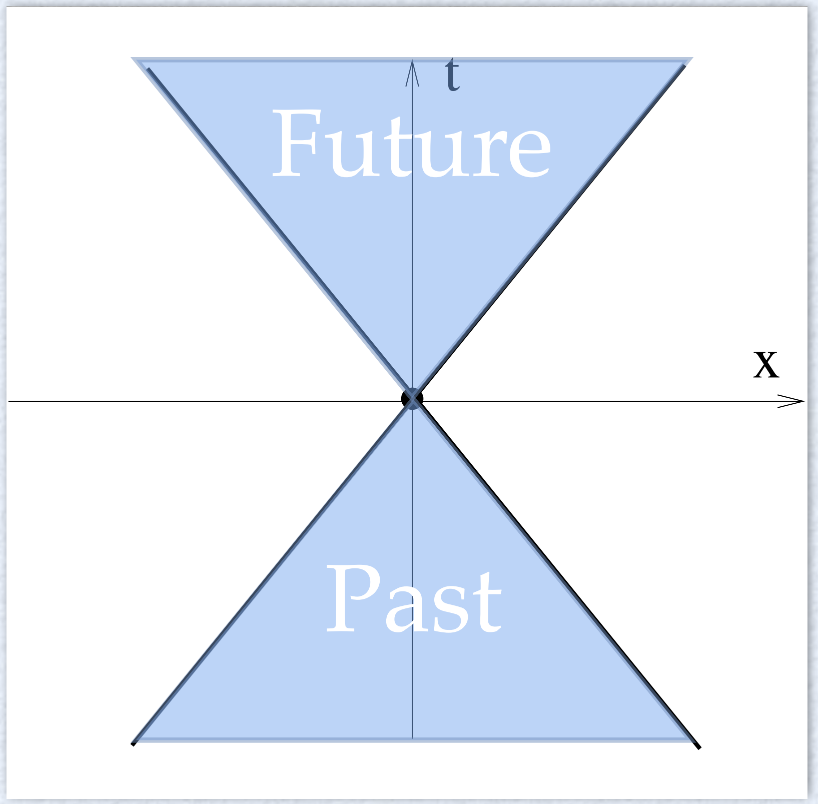
$x' = (x - vt)/\sqrt{1 - v^2}$
$t' = (t - vx)/\sqrt{1 - v^2}$
$y' = y$
$z' = z$

Einstein's Special Theory of Relativity published 1905

- Consistent theory of mechanics and electromagnetism
- Time itself is not absolute — it is observer dependent and our notion of simultaneity fundamentally changed

2.2 Special Relativity and Observers

Light cone divides space-time…



Coordinates are observer dependent but the invariant interval

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2 = dx'^2 + dy'^2 + dz'^2 - c^2\,dt'^2$$

is invariant. Note that light rays follow straight lines $ds^2 = 0$ and initially parallel trajectories remain so.

3.1 Equivalence and Mach

Newtonian gravity is *not* consistent with this picture, as the action of F is `at a distance' and instantaneous.

It would have been possible to emulate Maxwellian Electrodynamics, but Einstein took a different approach.

Equivalence is the observation that bodies in a gravitational field have the same space-time trajectory independent of their mass

$$F = \frac{G M m}{r^2} = m a$$

Gravitational mass = Inertial mass

3.2 Equivalence and Mach

Einstein interpreted this principle as the statement that

The gravitational influence of a massive body is to distort space-time itself

rather than to generate a field that reaches out over a fixed space-time. Furthermore

The trajectory of a body moving under the influence of gravity alone (ie in free-fall) is a geodesic of the space-time (ie is an extremum of the invariant interval)

This formulation is the essence of General Relativity. Whether it is `right' is ultimately an experimental and observational question.

3.3 Equvalence and Mach

Einstein coined the term 'Mach's Principle' in his discussion of one consequence that was immediately clear — the notion of an independent inertial observer has been lost…

- The structure of space-time everywhere is determined by what is there, including the observer itself
- Only by knowing the structure of space-time between observer and the observed can the observer correctly interpret what is seen

4.1 Metrics and Parallel Transport

To implement this theory we need to describe curved space-times. They are characterised by the metric g

$$ds^2 = g_{11}\, dx^2 + g_{22}\, dy^2 + g_{21}\, dx\, dy + \dots \quad + c^2 g_{00}\, dt^2$$

We need a more compact notation so define

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \quad \mu = 0,1,2,3$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu}\, dx^\mu dx^\nu \qquad \text{“Contraction”}$$

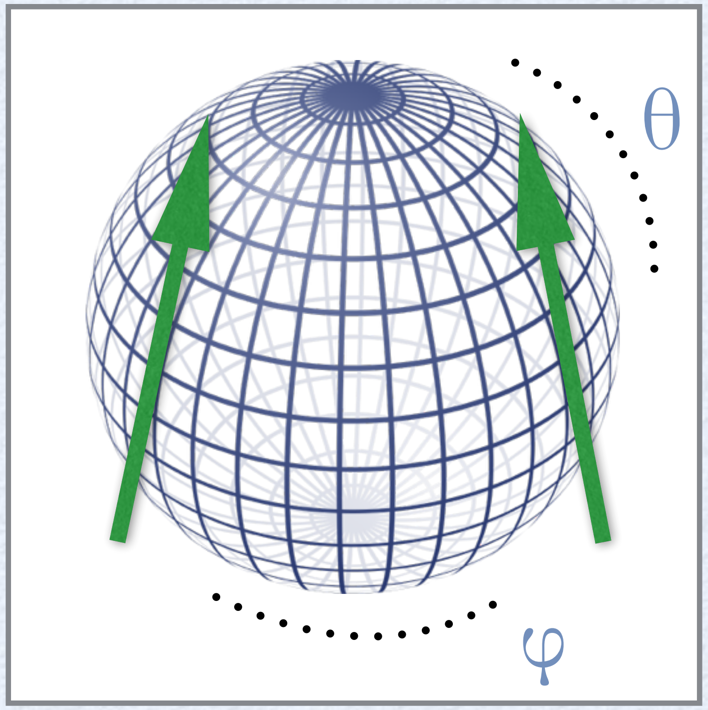
4.2 Metrics and Parallel Transport

For the flat space time in Cartesian coords we simply have

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

— the Minkowski metric

An example of curved space is

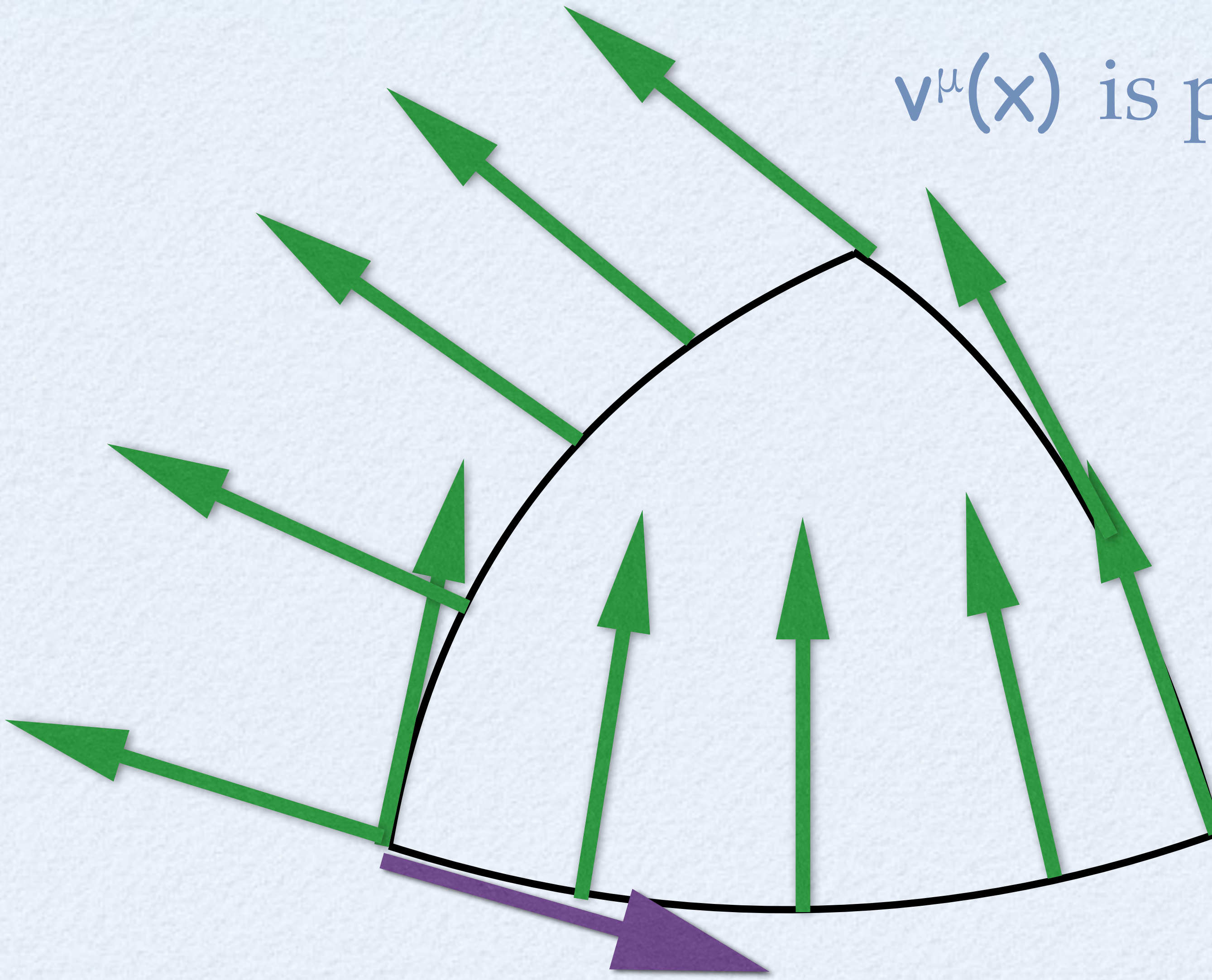


$$ds^2 = K^2 \; (d\theta^2 + \sin^2\theta \; d\varphi^2)$$

$$g_{\theta\theta} = K^2, \quad g_{\varphi\varphi} = K^2 \sin^2\theta, \quad g_{\varphi\theta} = g_{\theta\varphi} = 0$$

How do we know whether these two vectors are the same?

4.3 Metrics and parallel transport



$v^\mu(x)$ is parallel transported along t^ν

$$0 = t^\nu \nabla_\nu v^\mu$$

$$\nabla_\nu v^\mu = \frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\lambda} v^\lambda$$

Flat space

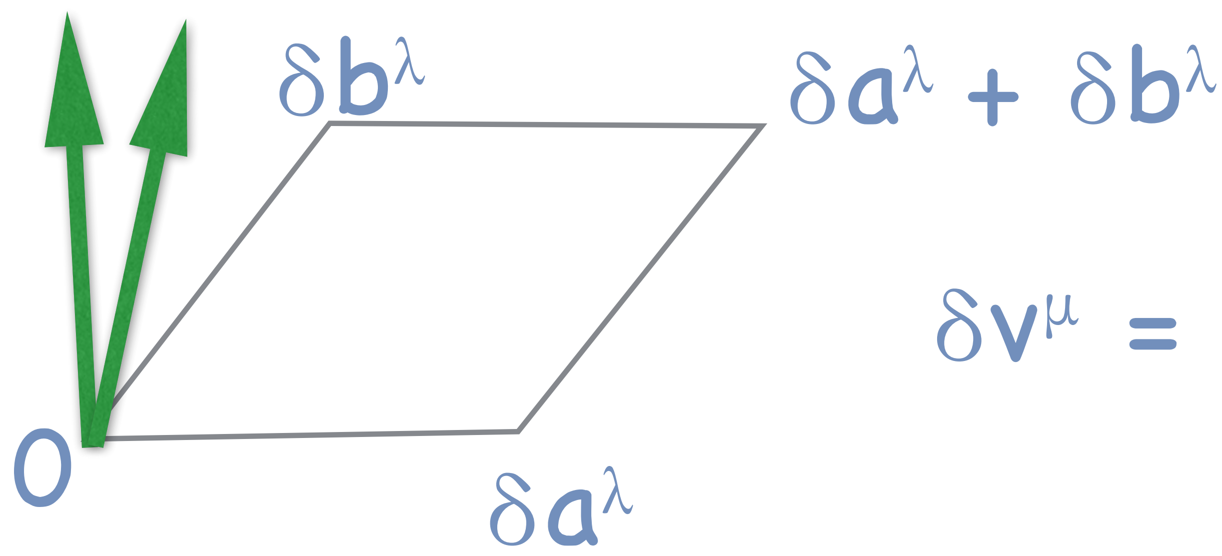
Connection (Christoffel)

$$2g_{\lambda\rho}\Gamma^\rho{}_{\mu\nu} = \frac{\partial g_{\lambda\nu}}{\partial x^\mu} + \frac{\partial g_{\lambda\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}$$

- NB Transport along different paths gives different results!

5.1 Curvature and Geodesics

The result of parallel transport round a *closed path* measures the *curvature*



Riemann Tensor

$$\delta v^{\mu} = \delta a^{\alpha} \delta b^{\beta} R_{\alpha\beta\lambda}{}^{\mu} v^{\lambda}$$

- $R_{\alpha\beta\lambda}{}^{\mu}$ has dimension L^{-2}
- $R_{\alpha\beta\lambda}{}^{\beta} = R_{\alpha\lambda}$ is called the Ricci Tensor
- $R_{\alpha\lambda} g^{\alpha\lambda} = R$ is called the Ricci Scalar
- $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu}$ is called the Einstein Tensor

5.2 Curvature and geodesics

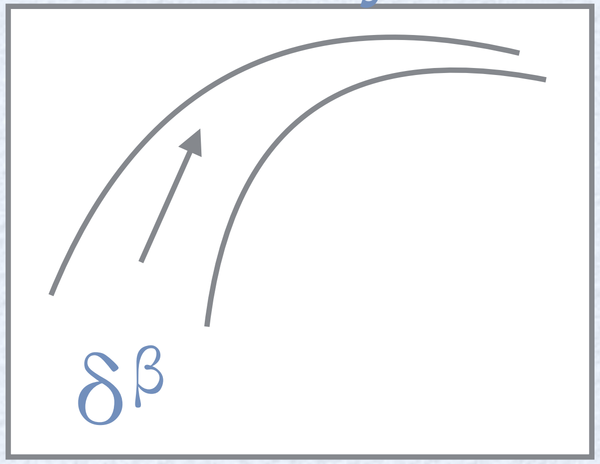
A *geodesic* is a curve whose tangent vector t^ν satisfies

$$0 = t^\nu \nabla_\nu t^\mu$$

This condition extremises the proper interval locally — it is the straightest possible path between two points. If the curve is $x^\mu(s)$ then

$$t^\mu = \frac{dx^\mu}{ds} \qquad \text{and so} \qquad \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0$$

Finally we can show that the separation δ^μ of nearby geodesics accelerates



$$\ddot{\delta}^\mu = -\,\delta^\beta R_{\alpha\beta\lambda}{}^\mu\, t^\alpha\, t^\lambda$$

6.1 Einstein's Equations

Einstein required a theory in which

- Gravitational Field (ie as an analogue of EM field) is replaced by space-time curvature
- Space-time curvature is generated by mass/energy
- The motion of particles under gravitational influence is described by the geodesics — these exist as properties of the space-time alone independently of the particle
- Special Relativistic physics is recovered as $R_{\mu\nu\lambda\rho} \rightarrow 0$, and Newtonian Gravity is recovered as $c \rightarrow \infty$

6.2 Einstein's Equations

We must have an equation of the form

curvature = constant x (Mass or Energy)

so that doubling the sources will double $R_{\alpha\beta\lambda}{}^{\mu}$ and double the effect on geodesics. The correct choice is not totally obvious and Einstein had some trouble with it…

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G c^{-4} T_{\mu\nu}$$

The form of the stress tensor $T_{\mu\nu}$ is very specific to the source and still poorly understood. We know which choices work - eg for a perfect fluid it is

$$T_{\mu\nu} = \rho\, u_{\mu}\, u_{\nu} + P\, (g_{\mu\nu} + u_{\mu}\, u_{\nu})$$

6.3 Einstein's Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G c^{-4} T_{\mu\nu}$$

The tensor form shows that this is an equation which is true in any coordinate system and thus rectifies the inconsistency of Newtonian gravity with Special Relativity.

Unfortunately the tensor form also masks very efficiently the horrendously complicated non-linear nature of the equations

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\mu \Gamma^\rho{}_{\rho\nu} + \Gamma^\rho{}_{\mu\nu} \Gamma^\lambda{}_{\rho\lambda} - \Gamma^\rho{}_{\lambda\mu} \Gamma^\lambda{}_{\rho\nu}, \quad \partial_\rho = \frac{\partial}{\partial x^\rho}$$

Remember $\Gamma^\rho{}_{\mu\nu}$ is a first derivative of g so Einsteins Equations are second order non-linear p.d.e.'s for g