Black holes in Einstein's gravity *and beyond*

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20 September 2014

Outline

Gravity and the metric

Einstein's equations

Symmetry yields solutions:
Schwarzschild, Reissner-Nordstrom, Kerr, Kerr-Newman metrics

Black hole horizon

Coordinate vs true singularities

Laws of black hole mechanics

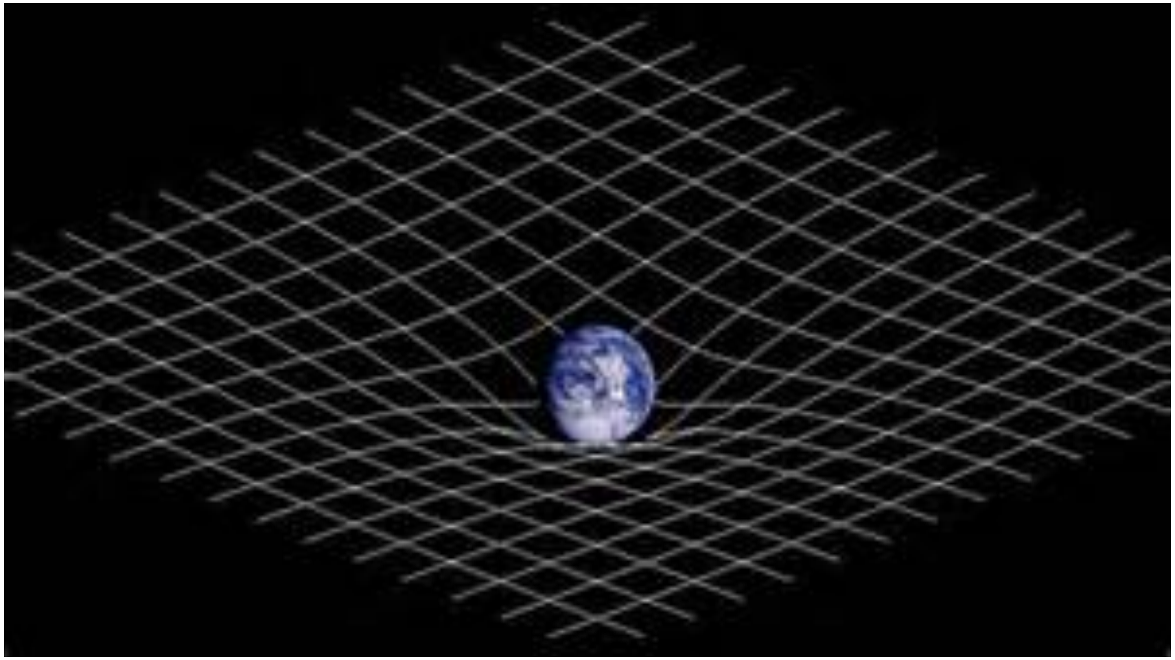
Do these laws reflect thermodynamics of a real system?

Hawking radiation and black hole entropy

Holographic principle

Gauge-gravity duality

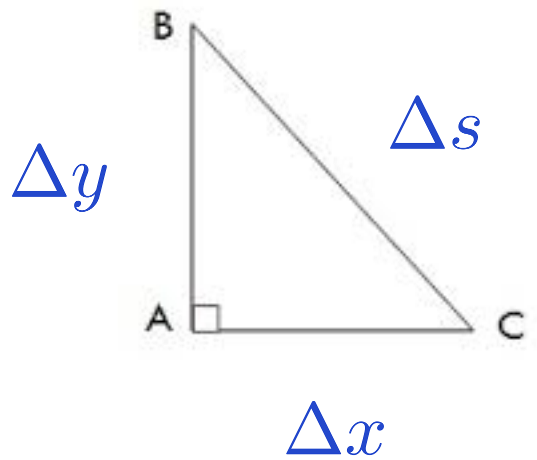
Gravity and the metric



$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Geometry (metric) = Energy (and/or mass)

Our Hero: The Metric



Distance in 2 dimensions: $\Delta s^2 = \Delta x^2 + \Delta y^2$

Infinitesimal distance in 2 dimensions: $ds^2 = dx^2 + dy^2$

Infinitesimal distance in 3 dimensions: $ds^2 = dx^2 + dy^2 + dz^2$

Our Hero: The Metric (continued)

Infinitesimal distance in D dimensions:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \cdots + (dx^D)^2$$

In matrix form in D=2 dimensions with $x^1 = x,\ x^2 = y$

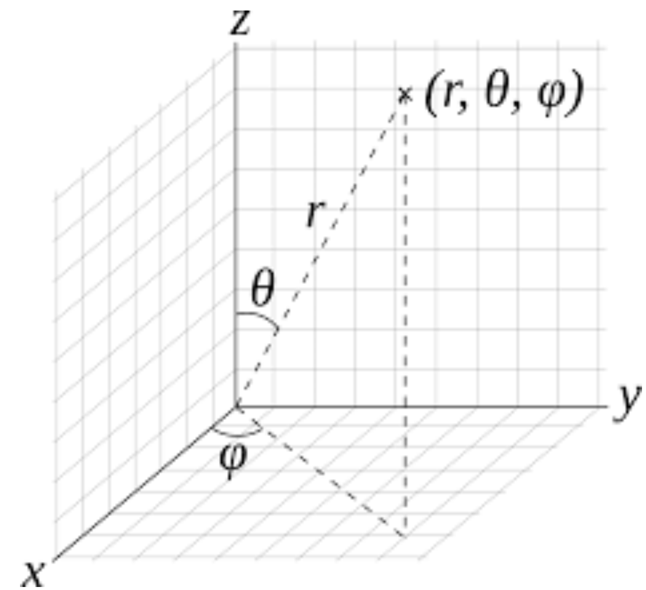
$$\begin{aligned} ds^2 = g_{ij}dx^i dx^j &= g_{11}dx^2 + g_{12}dxdy + g_{21}dydx + g_{22}dy^2 \\ &= dx^2 + dy^2 \end{aligned}$$

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The Metric Tensor

Infinitesimal distance in any dimension:

$$ds^2 = g_{ij}(x)dx^i dx^j$$



$g_{ij} = g_{ji}$ is a symmetric tensor

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$g_{ij}(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_{ij}(r,\theta,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The moral: different-looking metrics correspond to the same space

Examples of (flat) metrics

$$ds^2 = dx^2 + dy^2 + dz^2$$

Euclidean flat metric in 3 dimensions

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Minkowski (flat) metric in 3+1 – dimensional SPACE-TIME

$$ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

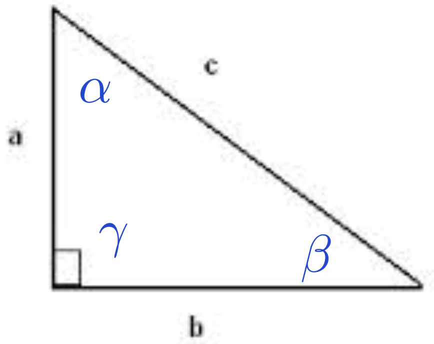
Minkowski (flat) metric in 3+1 – dimensional SPACE-TIME in spherical coordinates

How do we know the space is curved?

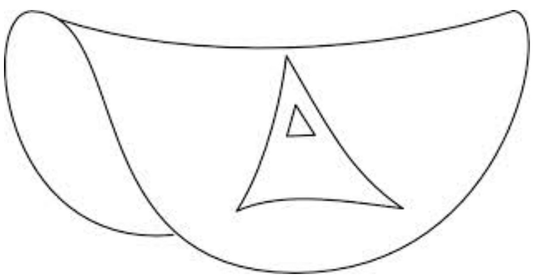
Carl Friedrich Gauss

Nikolai Ivanovich Lobachevsky

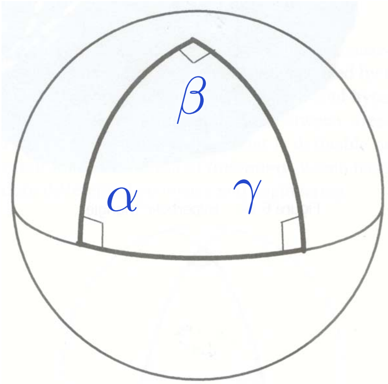
Janos Bolyai



$$\alpha + \beta + \gamma = 180^\circ$$

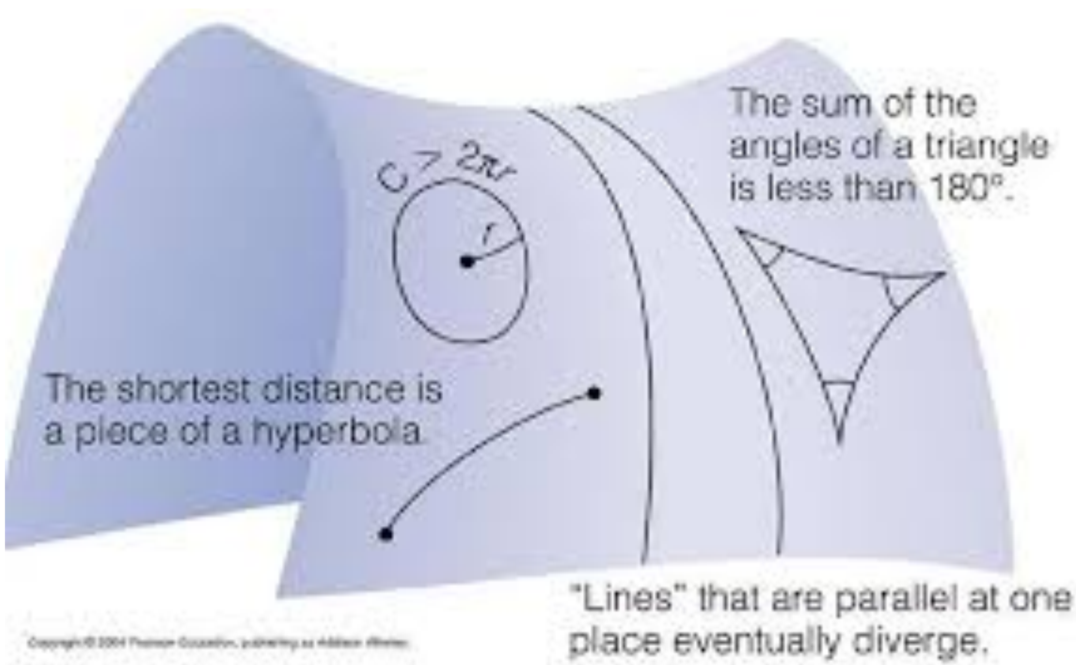


$$\alpha + \beta + \gamma < 180^\circ$$



$$\alpha + \beta + \gamma > 180^\circ$$

Hyperbolic space



$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

The Riemann tensor as the local measure of curvature

Bernhard Riemann

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^m_{ik}\,\Gamma^l_{mj} - \Gamma^m_{ij}\,\Gamma^l_{mk}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

$$R_{ij} = R^l_{ijl}$$ Ricci tensor

$$R = g^{ij}\, R_{ij}$$ Ricci scalar

A space is flat if and only if its Riemann tensor is zero

Einstein's equations

We now have all ingredients to write down Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

They are second-order non-linear coupled partial differential equations for the components of the metric tensor

The right hand side is a source (energy-momentum tensor)

$$\frac{\partial^2 g_{\mu\rho}}{\partial x^\rho \partial x^\nu} + \cdots = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Compare with Maxwell's equations:

$$\frac{\partial^2 A_\mu}{\partial x^\nu \partial x_\nu} + \cdots = -\mu_0 J_\mu$$

Solutions to Einstein's equations

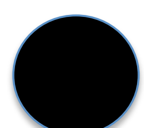
Even in a simpler case with zero cosmological constant and zero matter source

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

generic solution is unknown (recall Navier-Stokes equations) and is not unique

However, solutions corresponding to highly symmetric situations can be found by making relevant assumptions about the form of the metric

For example, if the source is spherically symmetric, we can assume the solution will be spherically symmetric, too



spherically—symmetric distribution of mass M

$$ds^2 = -A(r)c^2 dt^2 + B(r) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

Substitute into equations, solve for A and B (A=1,B=1 correspond to flat space-time)

The Schwarzschild metric

$$ds^2 = -c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Describes the metric of space-time OUTSIDE of a body of mass M

Corrections to “1” are very small for stars & planets

Note the (coordinate) singularity at

$$r = R_S = \frac{2GM}{c^2}$$

(the Schwarzschild radius)

Karl Schwarzschild (1873-1916)

Black holes

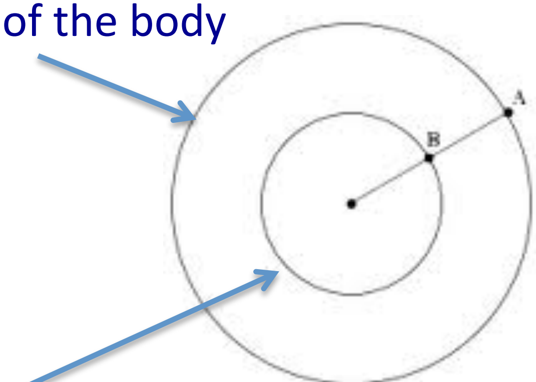
$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Describes the metric of space-time OUTSIDE of a body of mass M

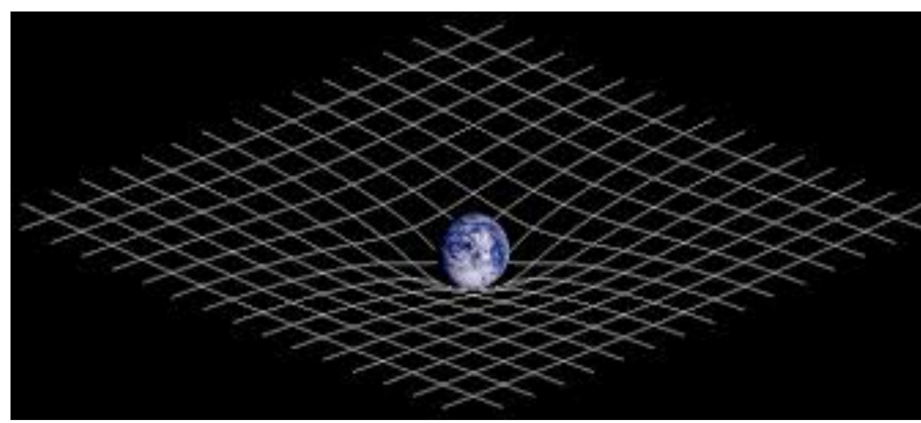
$$r = r_S = \frac{2GM}{c^2}$$

For most objects, the Schwarzschild radius is located deep inside the object and thus is not relevant since the solution is valid only outside (inside the metric is different). For example, for Earth the Schwarzschild radius is about 1 cm, for Sun – 3 km.

Surface of the body



Schwarzschild radius



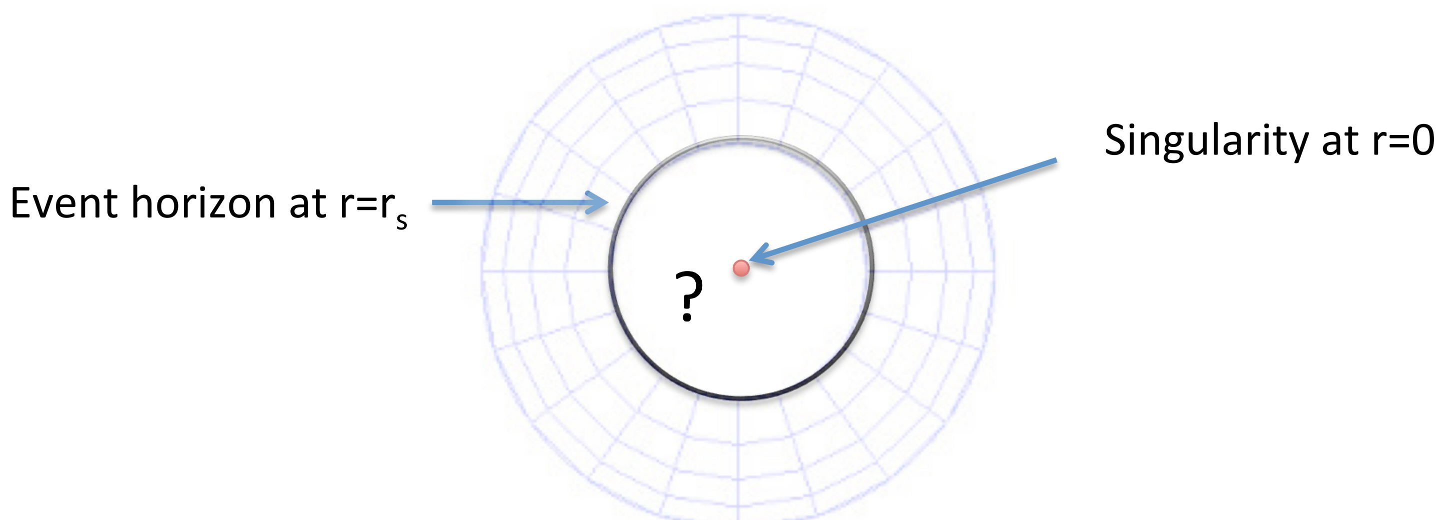
Black holes (continued)

$$ds^2 = -c^2\left(1-\frac{2GM}{c^2 r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{c^2 r}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Describes the metric of space-time OUTSIDE of a body of mass M

$$r = r_S = \frac{2GM}{c^2}$$

However, if the matter is squeezed inside its Schwarzschild radius (e.g. in the process of a gravitational collapse of a star), we get a black hole



Black holes: True and coordinate singularities

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Recall that the metric is a TENSOR = looks different in different coordinates

$$ds^2 = dx^2 + dy^2 \qquad\qquad ds^2 = dr^2 + r^2 d\varphi^2$$

Example: Cartesian vs polar coordinates - r=0 is a coordinate singularity

Similarly, one can write the Schwarzschild metric in different coordinates (e.g. Eddington-Finkelstein) where the metric is smooth at $r=r_s$

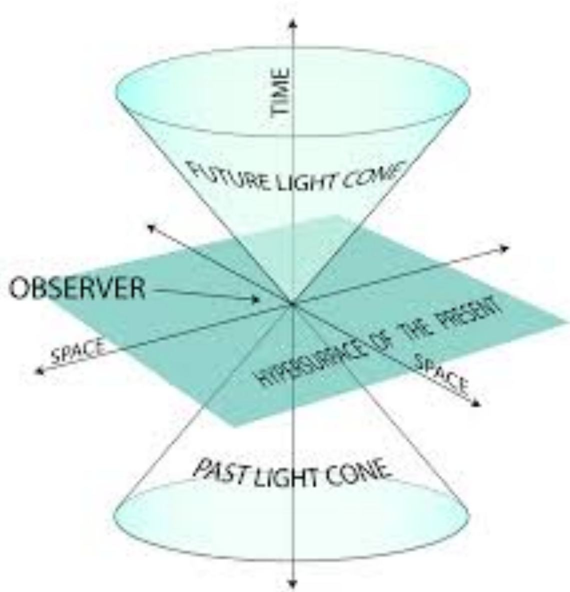
To check whether we have a true or coordinate singularity, need to compute something That stays invariant when coordinates are changed. For example, the Kretschmann invariant

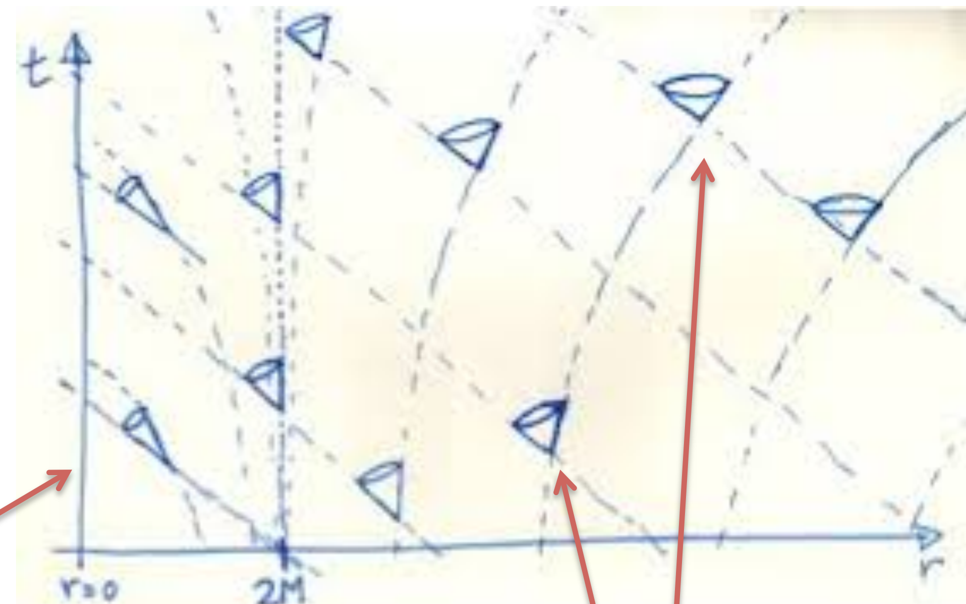
$$K = R_{ijkl}R^{ijkl} = \frac{48G^2M^2}{c^4 r^6}$$

The only true singularity is r=0

Black hole horizon

The fate of light cones





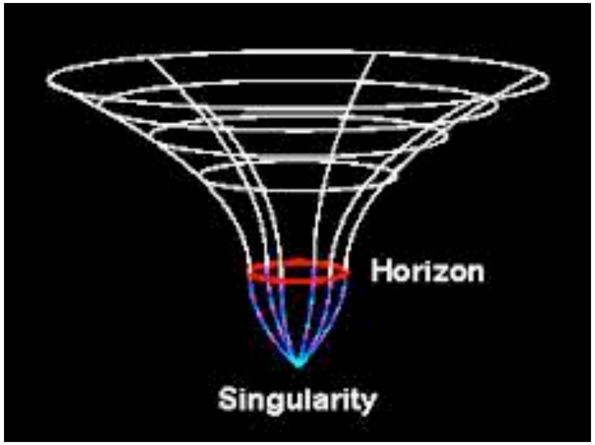
Singularity

Event horizon

Light cones outside the horizon

Black hole singularities
and the limits of modern physics

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$



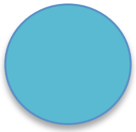
$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda + c_1 R^2 + c_2 R_{ij}R^{ij} + \cdots\right)$$

Corrections to the metric are expected near the singularity (quantum gravity, strings?)

Gravity as an effective theory at large distances & times

(difficult to test experimentally; no complete theory currently exists)

Symmetry yields solutions: Reissner-Nordstrom

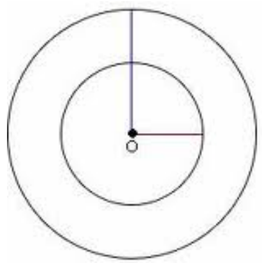


spherically—symmetric distribution of mass M and charge Q

Now we need to solve a system of Einstein-Maxwell equations to find a metric sourced by a charged non-rotating spherically symmetric body of mass M and charge Q

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_0 r^2}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\epsilon_0 r^2}\right)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

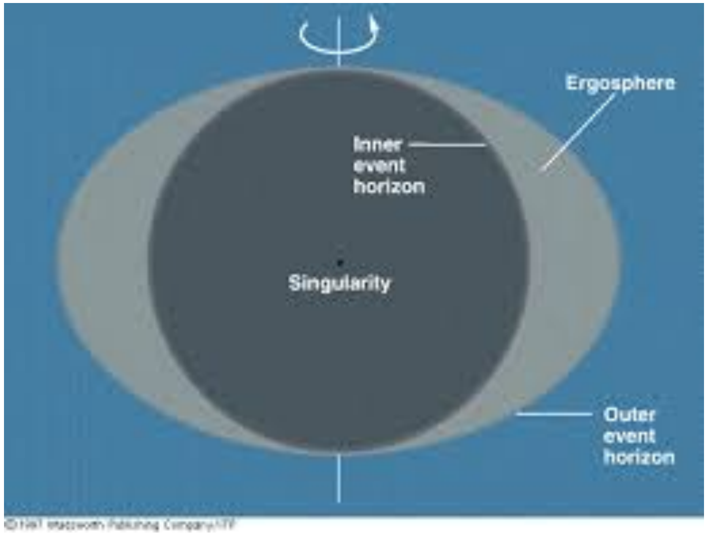
Reissner-Nordstrom solutions have two horizons



When the horizons coincide, the solution is called extremal (M=Q in relevant units)

Symmetry yields solutions: Kerr

A rotating axially-symmetric uncharged distribution of mass
Parameters: mass M and angular momentum J



Roy Kerr

Another exact solution (Kerr-Newman) describes
a rotating axially-symmetric charged distribution of mass
Parameters: mass M, charge Q and angular momentum J

The four laws of black hole mechanics

(J.Bardeen, B.Carter, S.Hawking, 1973)

Zeroth Law: The horizon of a stationary black hole has constant surface gravity κ

First Law: In perturbations of stationary black holes, the change of mass M is related to the change of charge Q, angular momentum J and horizon area A by

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

(here Ω is the angular velocity and Φ is the electric potential)

Second Law: The horizon area A is a non-decreasing function of time

Third Law: It is impossible to achieve zero surface gravity by a physical process

Laws of black hole mechanics vs laws of thermodynamics

Zeroth Law (BH): The horizon of a stationary black hole has constant surface gravity κ

Zeroth Law (TD): The system in thermal equilibrium has a constant temperature T

First Law (BH): In perturbations of stationary black holes, the change of mass M is related to the change of charge Q, angular momentum J and horizon area A by

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$$

First Law (TD): In thermodynamic processes the change of energy E is related to the change of entropy S (plus relevant work terms)

$$dE = TdS - p\,dV - \mu\,dN$$

Second Law (BH): The horizon area A is a non-decreasing function of time

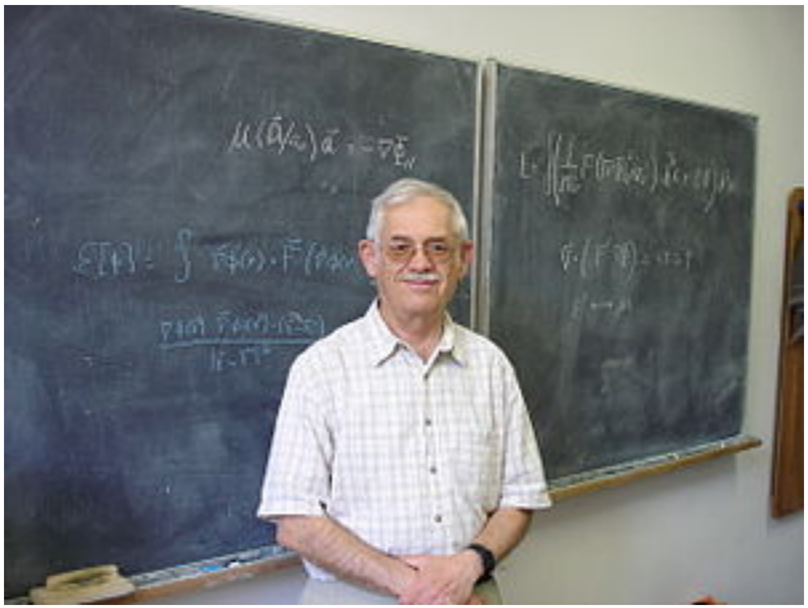
Second Law (TD): The entropy S is a non-decreasing function of time

Third Law: It is impossible to achieve zero surface gravity by a physical process

Third Law: It is impossible to achieve zero temperature by a physical process

Black hole thermodynamics

In 1972, Jacob Bekenstein suggested that a black hole should have a well defined entropy proportional to the horizon area. He was laughed at by some (many…) since black holes are BLACK (do not radiate) and thus cannot have a temperature associated with them



Jacob Bekenstein

Stephen Hawking

However, in 1974 Hawking demonstrated that black holes do emit radiation at a quantum level and so one can in fact associate a temperature with them

Hawking temperature and Bekenstein-Hawking entropy

Hawking showed that black holes emit radiation with a black-body spectrum at a temperature

$$T = \frac{\hbar c^3}{8\pi k_B G M} \approx \frac{1.2 \times 10^{23}\,\mathrm{kg}}{M}\, K$$

(a black hole of one solar mass has a Hawking temperature of about 50 nanoKelvin)

This fixes the coefficient of proportionality in Bekenstein's conjecture:

$$S_{BH} = \frac{c^3 A}{4 G \hbar}$$

Immediate consequences and problems:

- Black holes "evaporate" with time
- Information loss paradox
- What are the microscopic degrees of freedom underlying the BH thermodynamics?

Entropy and microstates

In statistical mechanics, entropy is related to the number of microstates:

N=1 particle with spin: $\downarrow \quad \uparrow$ 2 microstates

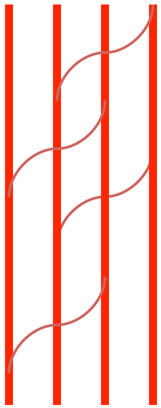
N=2 particles with spin: $\downarrow\downarrow \quad \downarrow\uparrow \quad \uparrow\downarrow \quad \uparrow\uparrow$ 4 microstates

N particles with spin: 2^N microstates

Boltzmann entropy: $$S = k_B \ln W = k_B \ln 2^N = k_B N \ln 2$$

Can we count the microstates of a black hole and recover the Bekenstein-Hawking result?

Counting microstates of a black hole



A.Strominger and C.Vafa (1996) were able to count the microstates of a very special (supersymmetric) black hole in 5 dimensions. The result coincides EXACTLY with the Bekenstein-Hawking thermodynamic entropy.

Strominger-Vafa result has been generalized in many ways since 1996.

However, we still do not know how to count the microstates of “normal” BHs, e.g. Schwarzschild BH in four dimensions…

Holographic principle

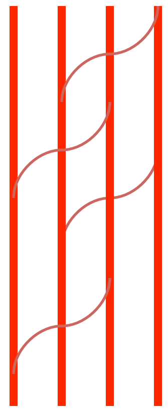
In thermodynamic systems without gravity, the entropy is extensive (proportional to volume)

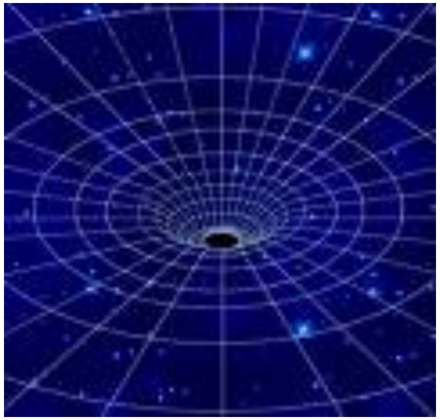
In gravitational systems, it is proportional to the AREA

$$S_{BH} = \frac{c^3 A}{4G\hbar}$$

It seems that gravitational degrees of freedom in D dimensions are effectively described by a theory in D-1 dimensions ('tHooft, Susskind, 1992)

Gauge-string duality (AdS-CFT correspondence)





Open strings picture:
dynamics of strings & branes at low energy
is desribed by a
quantum field theory without gravity

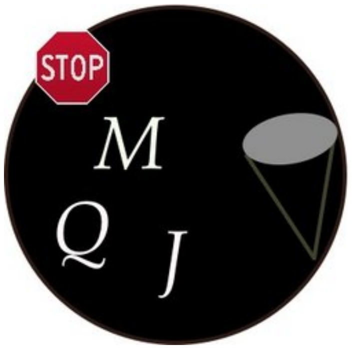
Closed strings picture: dynamics of strings
& branes at low energy is described by
gravity and other fields
in higher dimensions

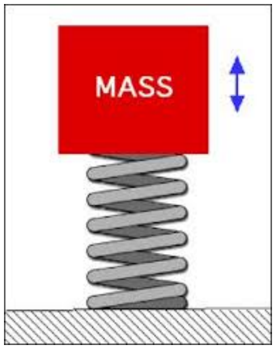
conjectured
exact equivalence

Maldacena (1997); Gubser, Klebanov, Polyakov (1998); Witten (1998)

Black holes beyond equilibrium

Undisturbed black holes are characterized by global charges: M, Q, J…





A thermodynamic system in equilibrium is characterized by conserved charges: E, Q, J…

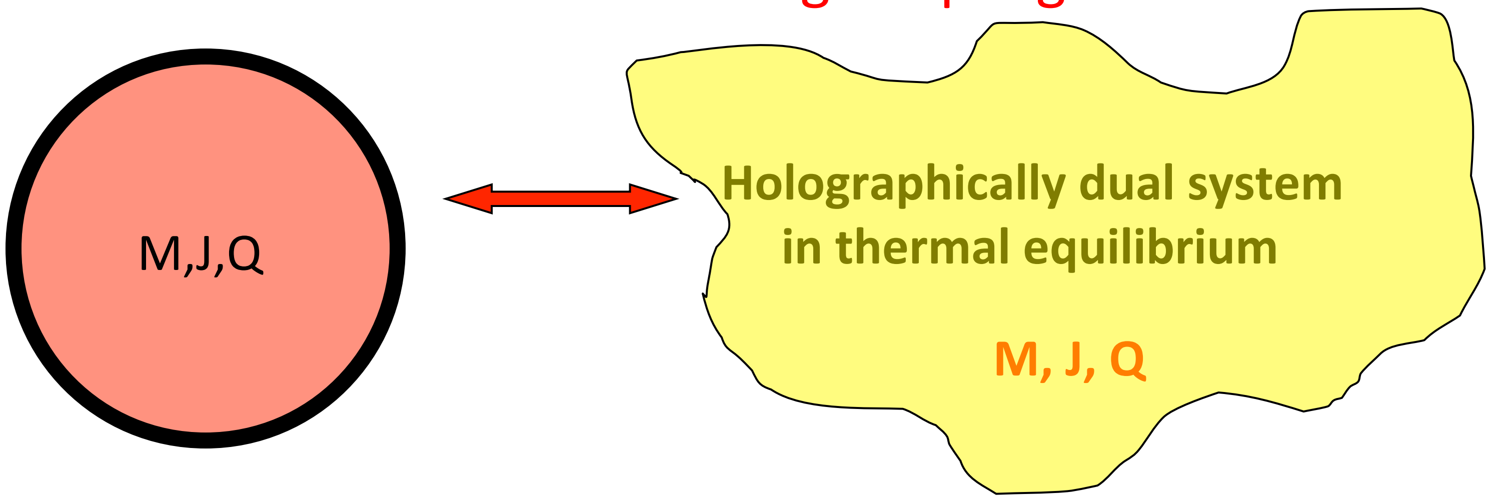
If one perturbs a non-gravitational system (e.g. a spring pendulum), it will oscillate with eigenfrequencies (normal modes) characterizing the system

$$\omega = \sqrt{\frac{k}{M}}$$

What happens if one perturbs a black hole?

10-dim gravity

4-dim gauge theory – large N, strong coupling



T_{Hawking} $S_{\text{Bekenstein-Hawking}}$ $\Longleftrightarrow$ T S

Gravitational fluctuations $\Longleftrightarrow$ Deviations from equilibrium

$g^{(0)}_{\mu\nu} + h_{\mu\nu}$ $\Longleftrightarrow$????

$$j_i = -D\partial_i j^0 + \cdots$$

$$\text{"}\Box\text{"}\ h_{\mu\nu} = 0$$ and B.C. $\Longleftrightarrow$

$$\partial_t j^0 + \partial_i j^i = 0$$

$$\partial_t j^0 = D\nabla^2 j^0$$

Quasinormal spectrum $\Longleftrightarrow$ $\omega = -iDq^2 + \cdots$

Black hole's quasinormal spectrum encodes properties of a dual microscopic system

Comparing eigenfrequencies of a black hole

$$\omega = \pm \frac{c}{\sqrt{3}} k - \frac{i}{6\pi T} k^2 + \frac{3 - 2\ln 2}{24\pi^2 \sqrt{3} T^2} k^3 + \cdots$$

with eigenfrequencies of a dual microscopic system (described by fluid mechanics)

$$\omega = \pm v_s k - i \frac{2\eta}{3sT} k^2 + \cdots$$

one can compute viscosity-entropy ratio and other quantities of a microscopic system

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} + \cdots$$

Moreover, one can relate Navier-Stokes equations and Einstein's equations…

Conclusions

Black holes are fascinating objects from the theoretical point of view

They test the limits of our knowledge:
Any decent candidate for a theory of quantum gravity must be able to explain their properties

Black holes have entropy and temperature and behave like thermodynamic systems,
and we think we know why (holographic principle)

Black hole spectra of excitations encode non-equlibrium properties of a dual microsystem

We do not know how to resolve the BH singularity

Einstein's gravity is an approximation, an effective theory…
We do not know the full theory at quantum level

We should not forget about practical applications and REAL Black hole in our very own Universe! Talks by John Magorrian and James Binney - NEXT…

Landau & Lifschitz, “Field Theory”…

THANK YOU!