Plasmas: The Normal Form of Matter and the Key to Unlimited Energy

Lecture 2. TURBULENCE: PLASMA UNLEASHED

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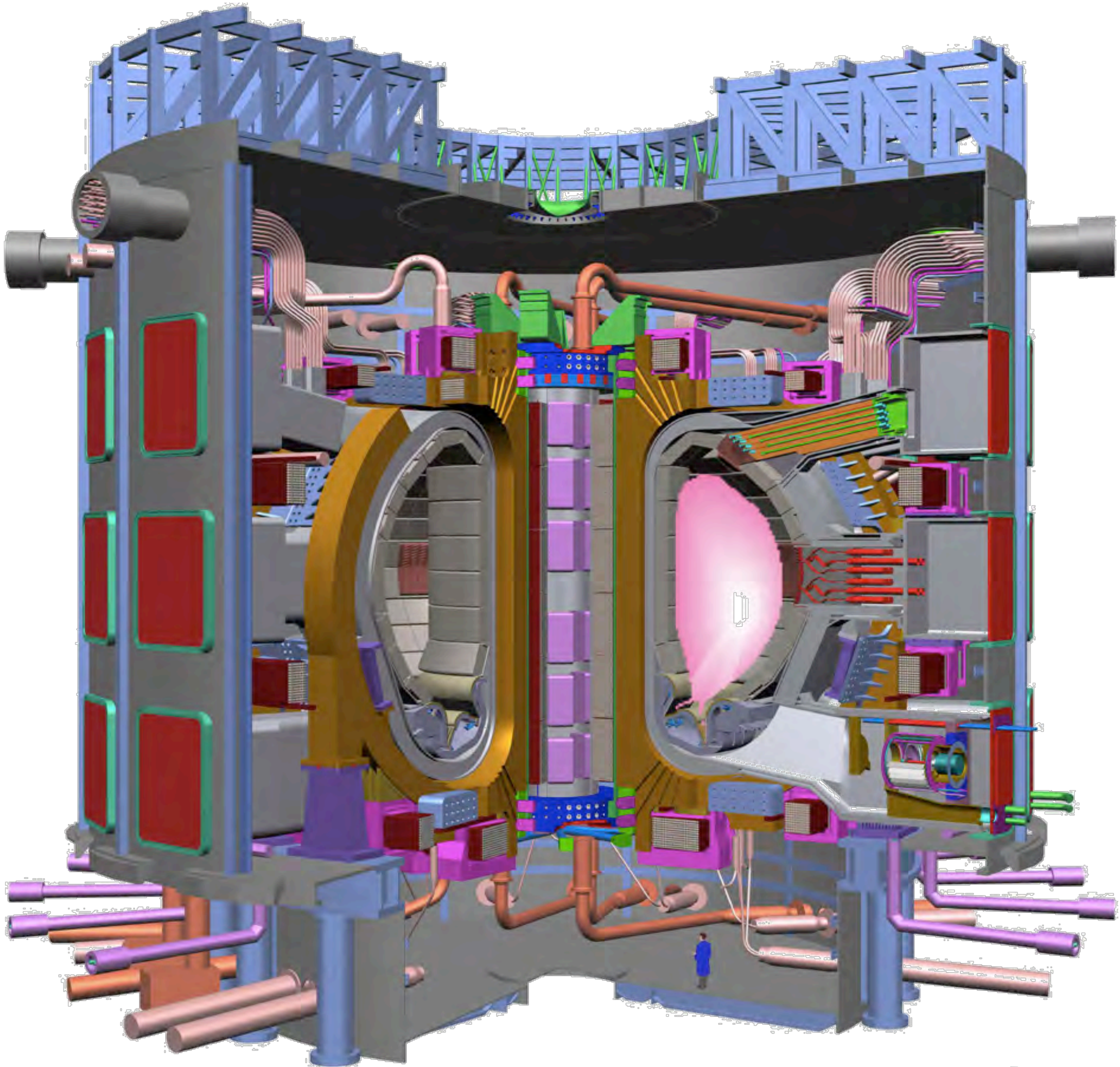
(The Rudolf Peierls Centre for Theoretical Physics
& Merton College)

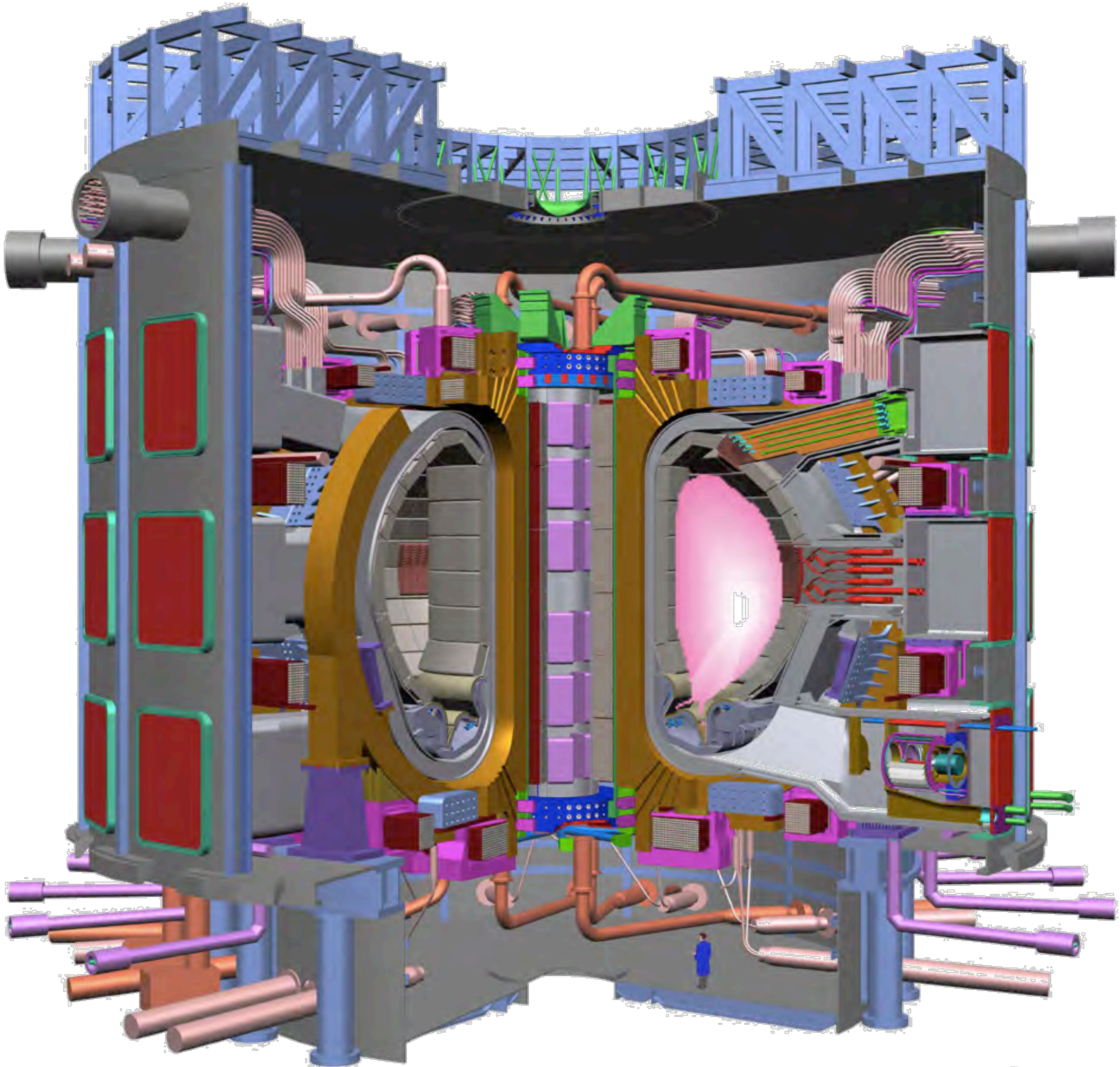
Thanks to the plasma group at Oxford/Culham: J. Ball, M. Barnes, G. Colyer, J. Connor FRS, S. Cowley FRS, P. Dellar, W. Dorland, A. Field, M. Fox, J. Hastie, E. Highcock, J. Hillesheim, A. Mallet, S. Melville, J. Parker, F. Parra, C. Roach, B. Taylor FRS, F. van Wyk

The Machine



[Image: ITER]

Find Theoretical Physics Here!



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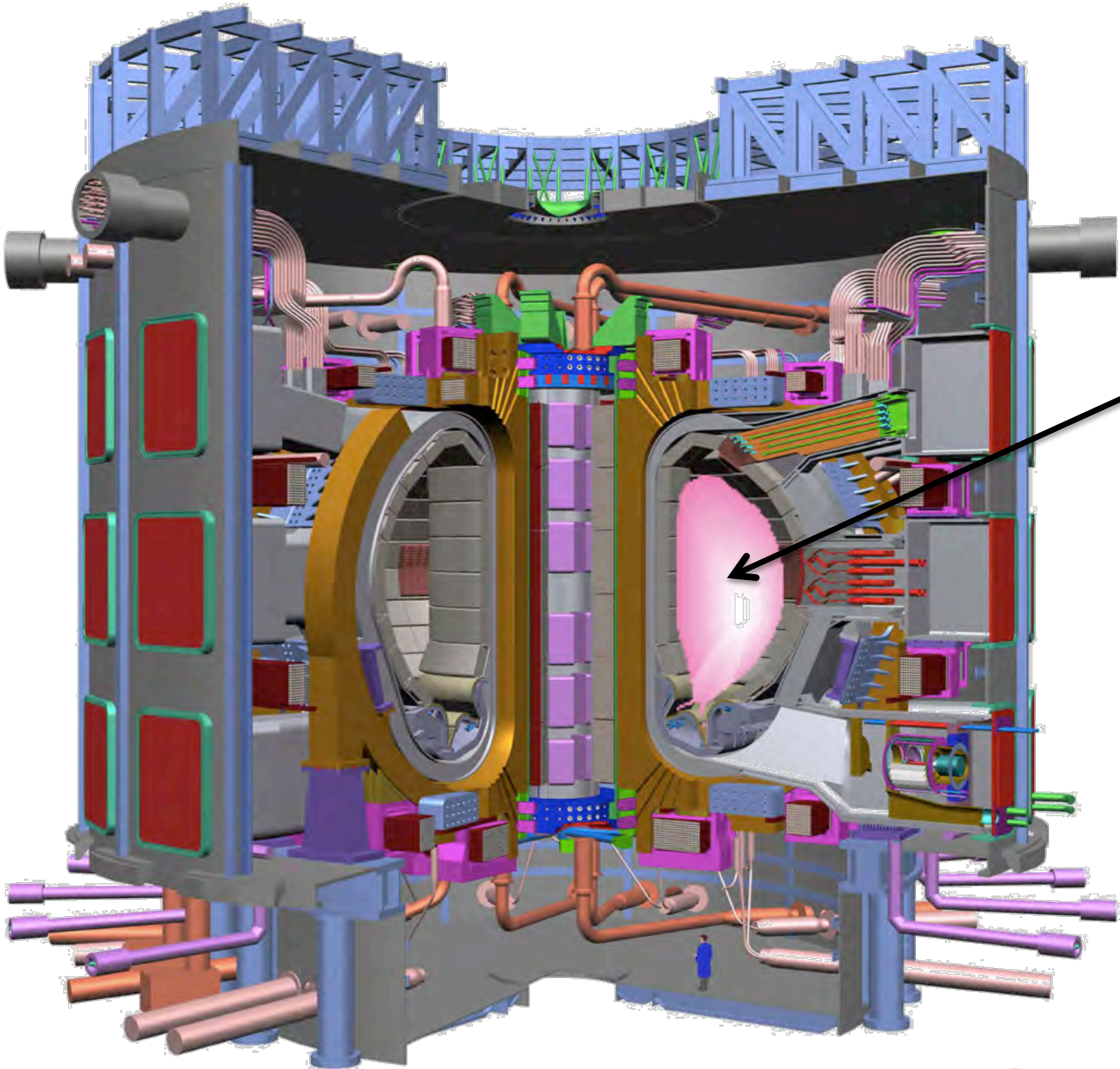

The Beast

[Image: ITER]

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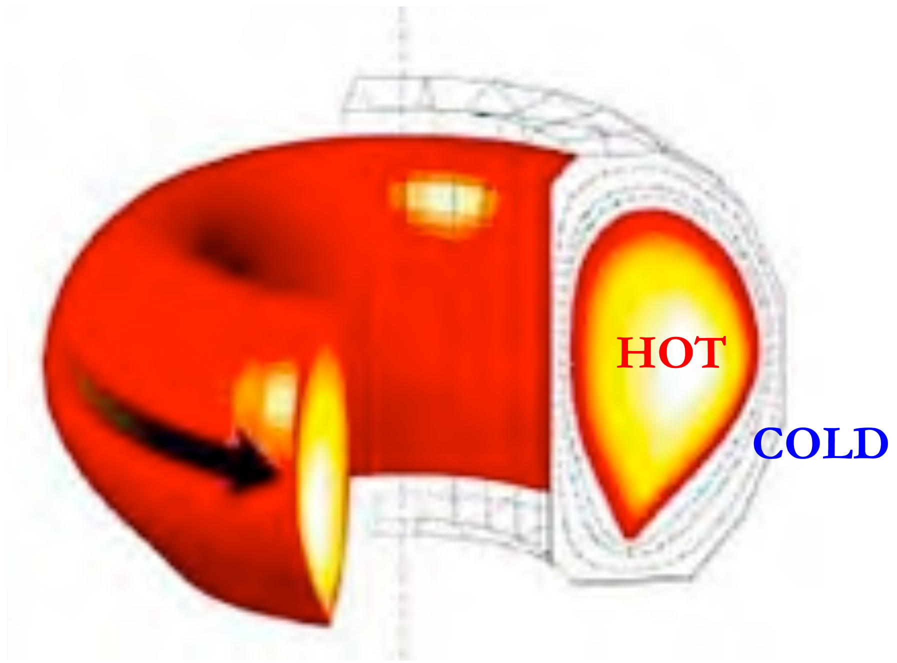

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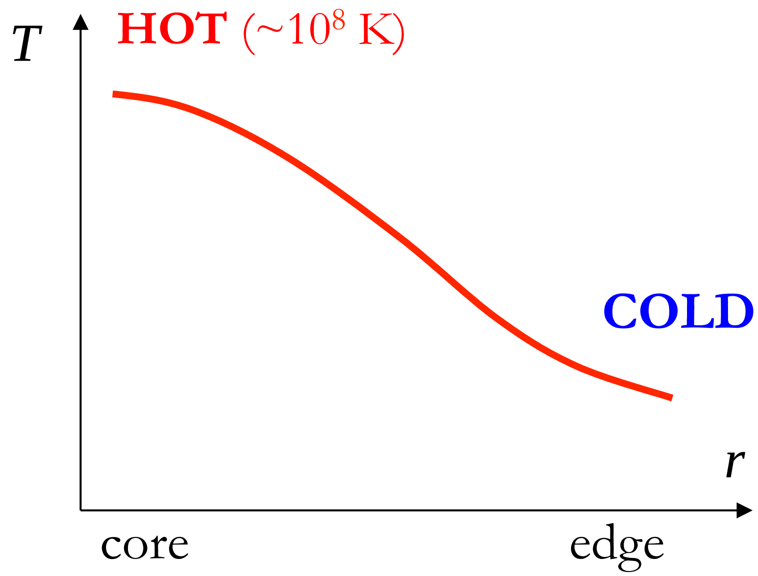
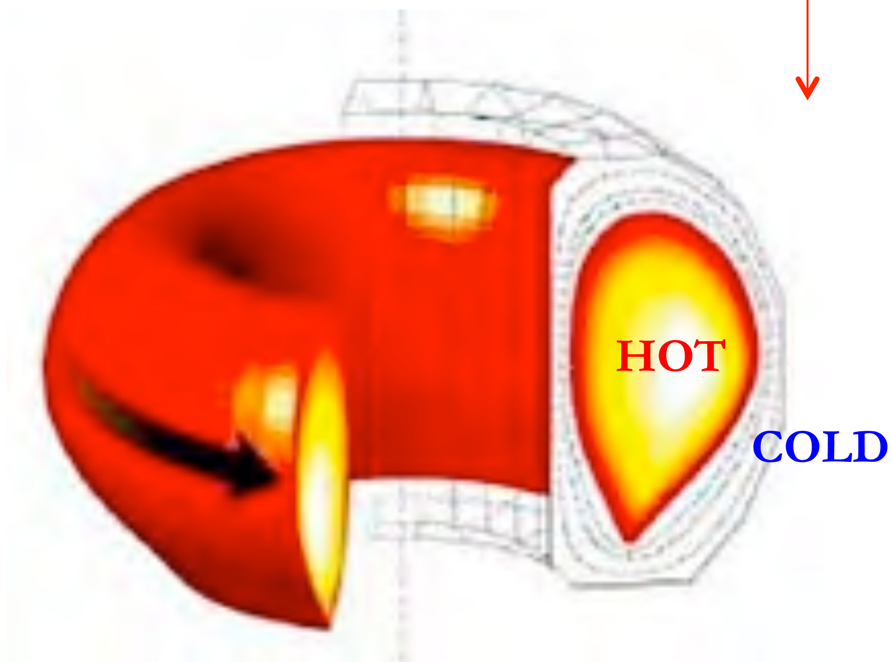


How hot does it get?
(How hot can we make it?)



Undergraduate Physics: Heat Transport

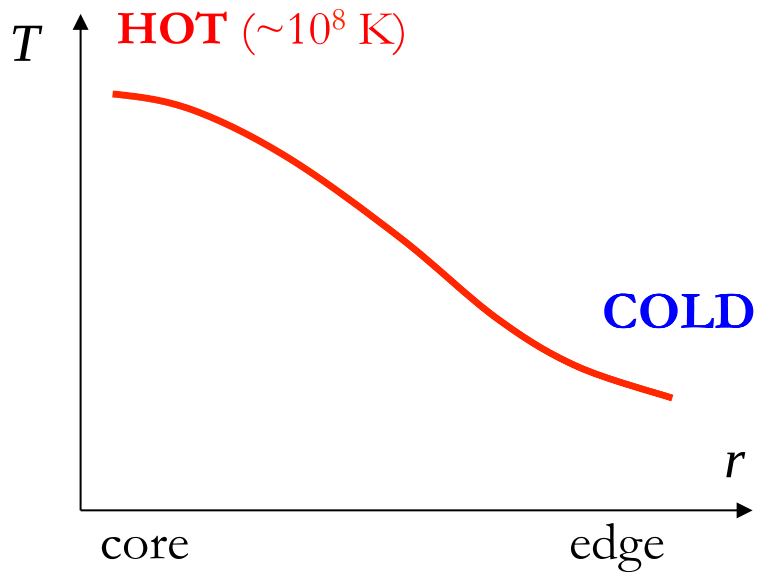
How hot does it get?
(How hot can we make it?)

2nd-year UG physics: heat equation

$$\frac{\partial T}{\partial t} = D\Delta T + S$$

heat diffusivity

sources – sinks (heating – cooling)



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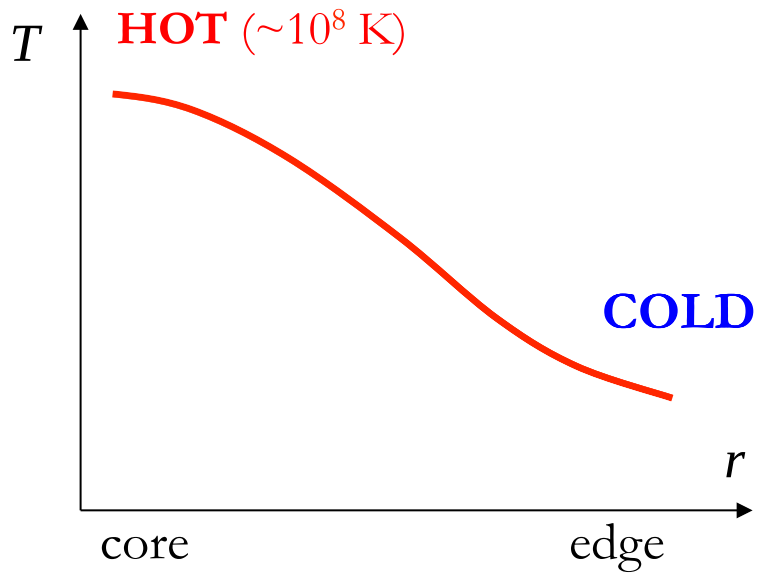
$\boxed{D\Delta T + S = 0}$ in steady state



Now solve this in a torus, knowing S and boundary conditions,
get temperature profile $T(r)$, hand solution over to engineers,
move on to thinking of dark matter, quantum entanglement, the brief history of time, etc…

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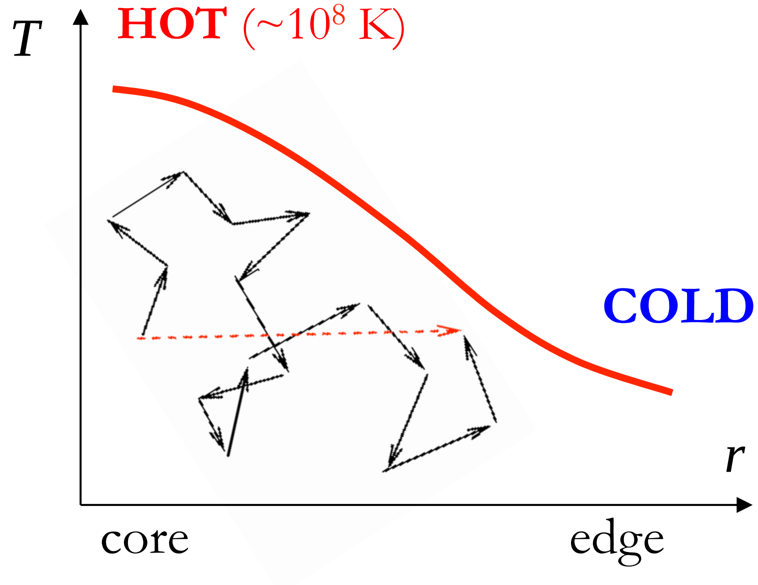

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Before we do that, what's D? It's a diffusion coefficient:

$$D \sim \frac{\langle \Delta x^2 \rangle}{\Delta t} \sim \langle v^2 \rangle \Delta t \sim c_s \lambda_{\mathrm{mfp}}$$ standard UG estimate

speed of sound
$v \sim c_s \sim (p/\rho)^{1/2}$

mean free path
between collisions

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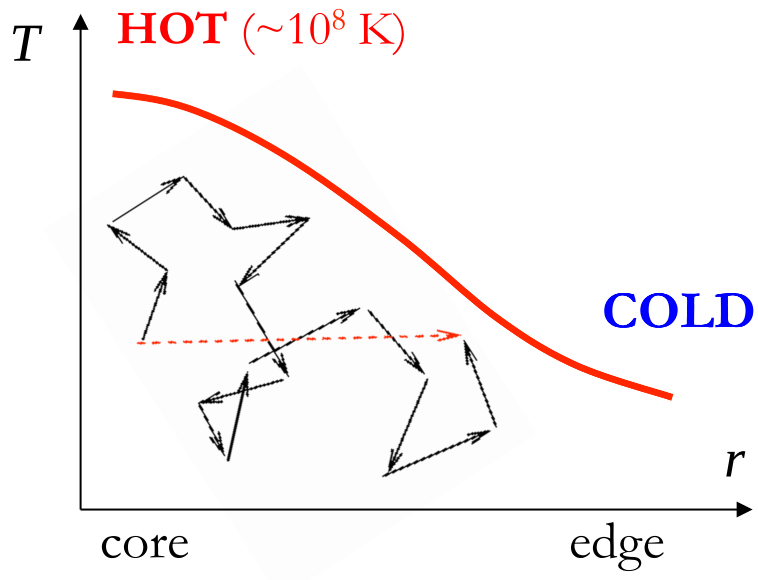

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Before we do that, what's *D?* It's a diffusion coefficient:

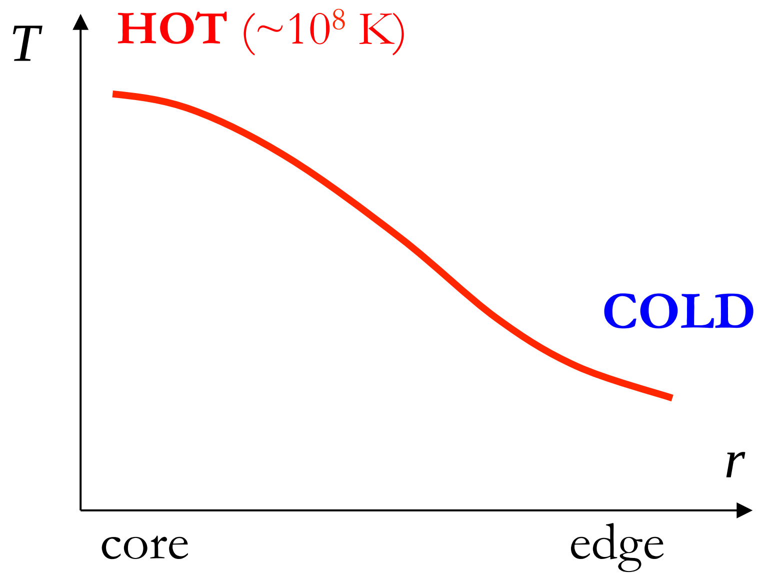
$$D \sim \frac{\langle \Delta x^2 \rangle}{\Delta t} \sim \frac{\rho_i^2}{\tau_c}$$ in a magnetised plasma

time between collisions
$\tau_c \sim \lambda_{mfp}/c_s$

Larmor radius of the ions
$\rho_i \sim c_s/\Omega_i$
$\Omega_i = eB/m_i c$

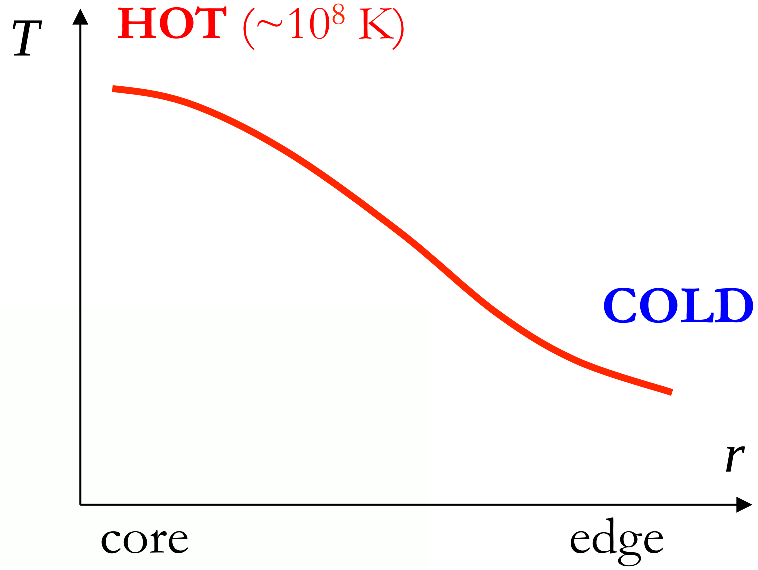
TOO SMALL TO EXPLAIN OBSERVED TRANSPORT!

Look Closer…



Look Closer…

Plasma in a tokamak is **turbulent** (nature dislikes gradients – lack of equilibrium! – and contrives to drive the system unstable)



Gyrokinetic simulation of the DIIID tokamak [R. Waltz & J. Candy, GA, San Diego]

Heat Diffusion + Turbulence

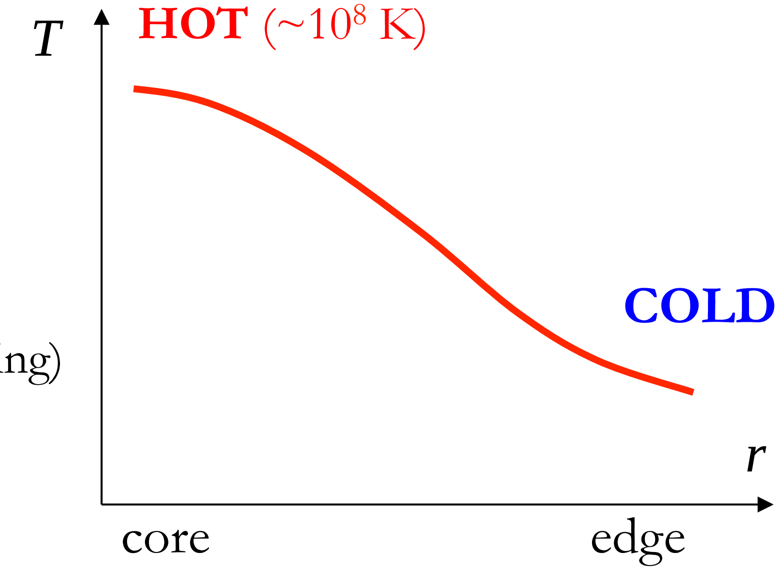
Heat equation in a moving medium:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = D \Delta T + S$$

velocity of plasma motions (chaotic!)

heat diffusivity

sources – sinks (heating – cooling)



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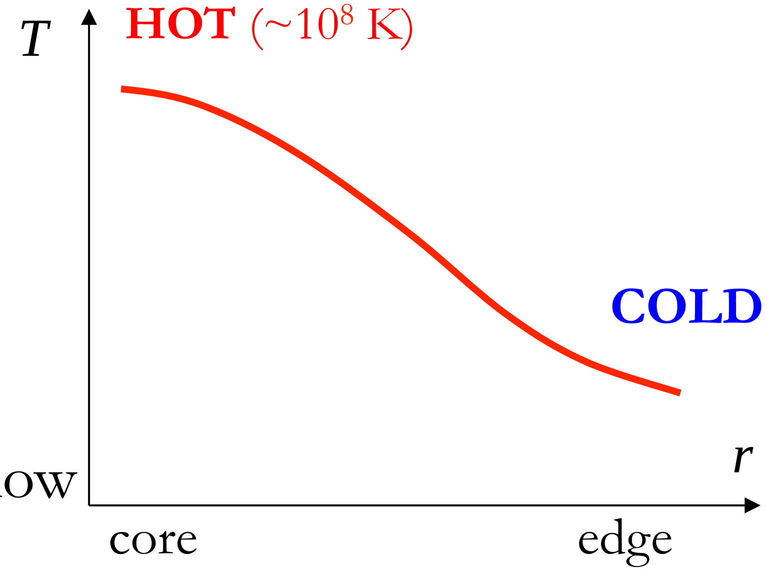
Mean profile: $\overline{T}(r) = \langle T \rangle$

$T = \overline{T} + \delta T\,,\ \delta T \ll \overline{T}\,,$

fluctuations are fast, mean quantities slow



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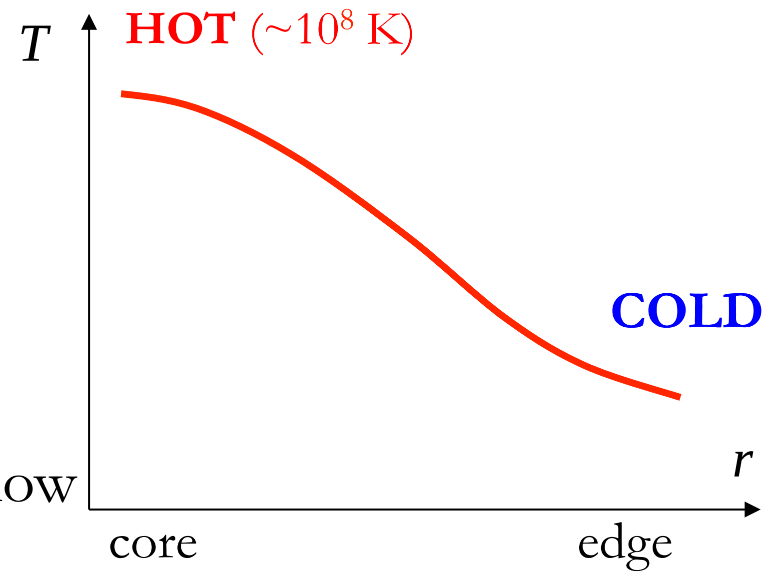

Then, assuming $\nabla \cdot \mathbf{u} = 0$,

$$\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \langle \mathbf{u}T \rangle = D\Delta\overline{T} + S$$

We need to know about fluctuations because

$\langle \mathbf{u}T \rangle = \langle \mathbf{u} \rangle \overline{T} + \langle \mathbf{u}\delta T \rangle = \langle \mathbf{u}\delta T \rangle$, assuming for now $\langle \mathbf{u} \rangle = 0$

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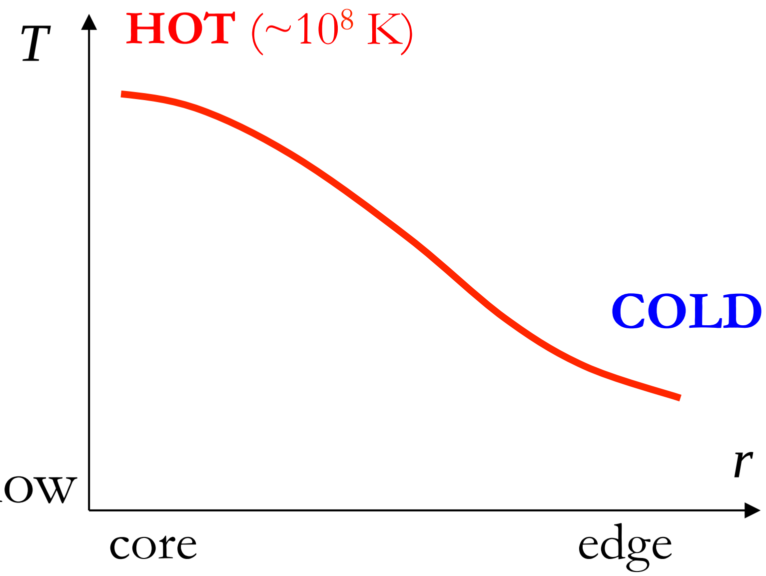

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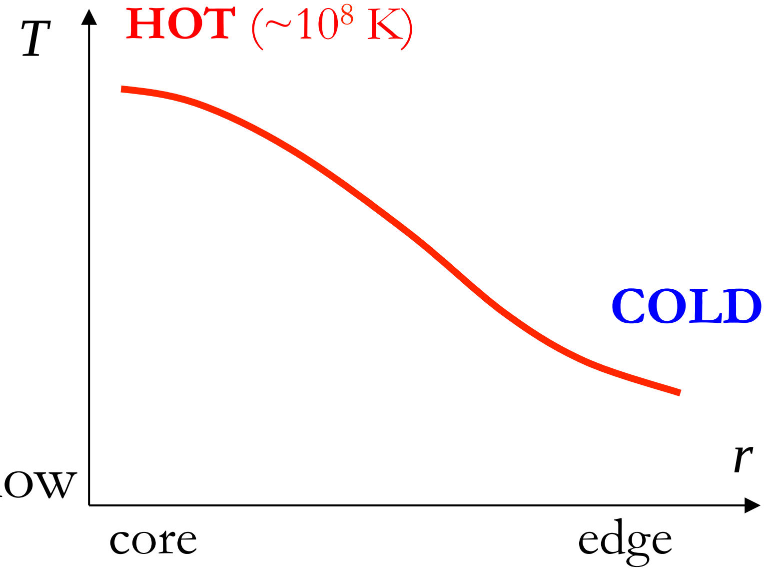

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$$\langle\mathbf{u}(t)T(t)\rangle \approx -\left[\int_0^t dt'\,\langle\mathbf{u}(t)\mathbf{u}(t')\rangle\right]\cdot\nabla\overline{T}(t)$$ "turbulent heat flux"

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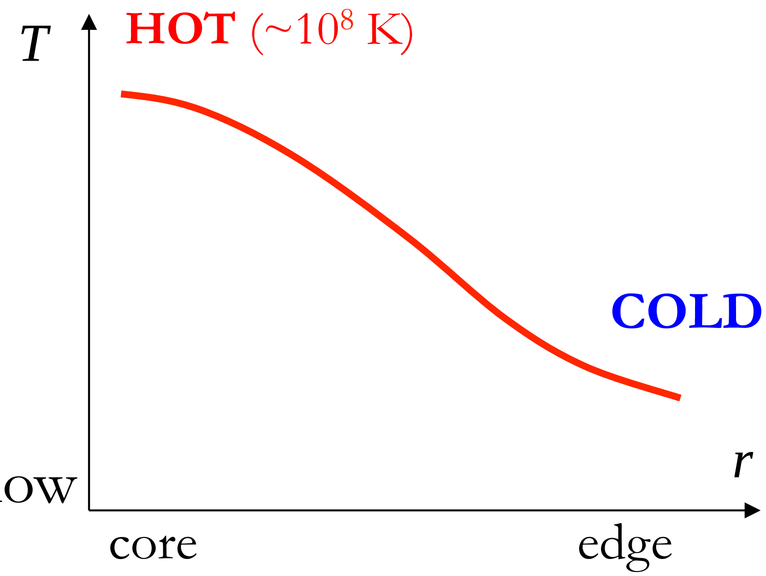

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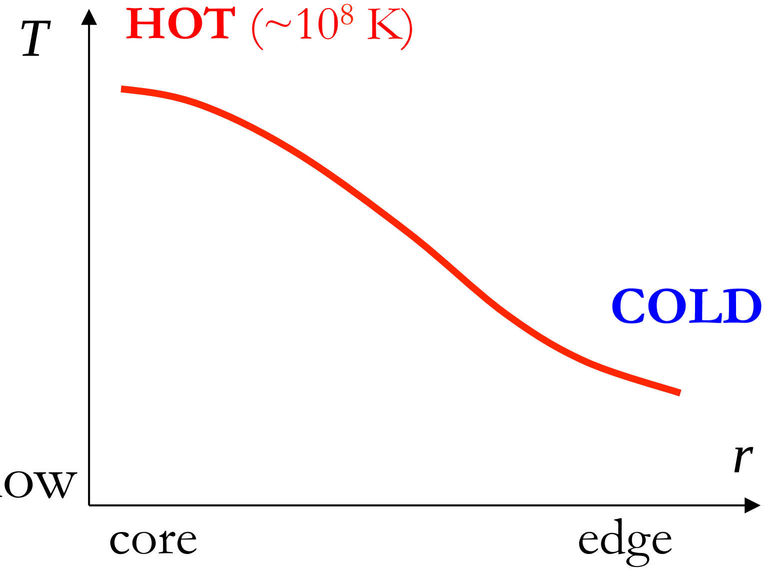

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$$= \frac{1}{r}\frac{\partial}{\partial r} r\left[D^{(\mathrm{turb})} + D\right]\frac{\partial \overline{T}}{\partial r} + S$$

"turbulent diffusion" $D^{(\mathrm{turb})} = \int_0^t dt'\langle u_r(t)u_r(t')\rangle$

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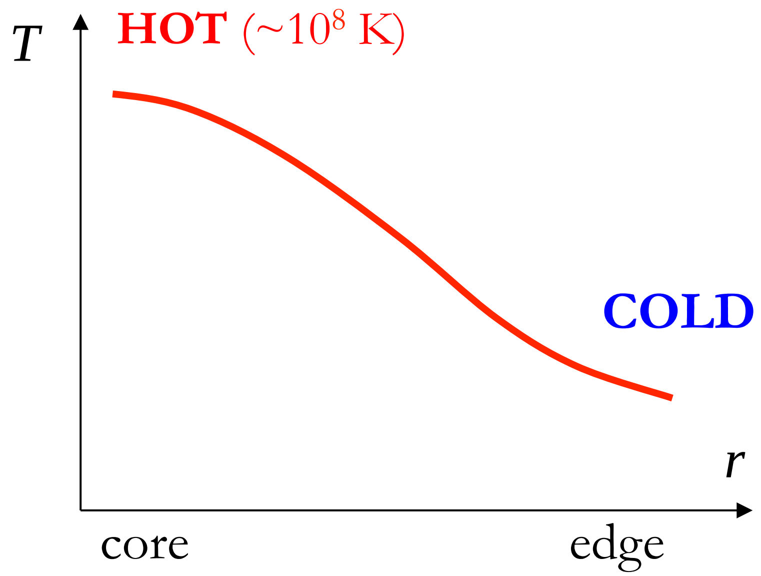

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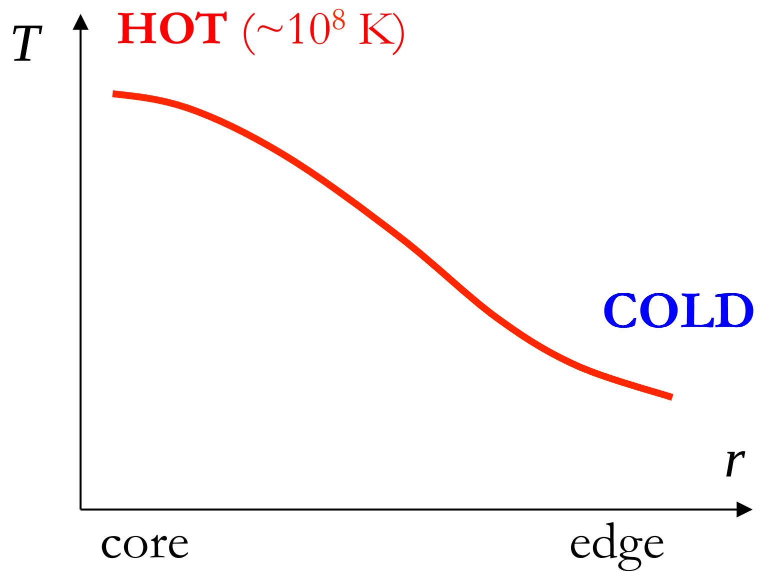

These ideas are universal: e.g., if you are a (plasma) astrophysicist, you know that the largest plasma objects are clusters of galaxies (containing mostly dark matter and hot, diffuse plasma, not galaxies):

(Abell 262
in optical,
http://www.atlasoftheuniverse.com/superc/perpsc.html)

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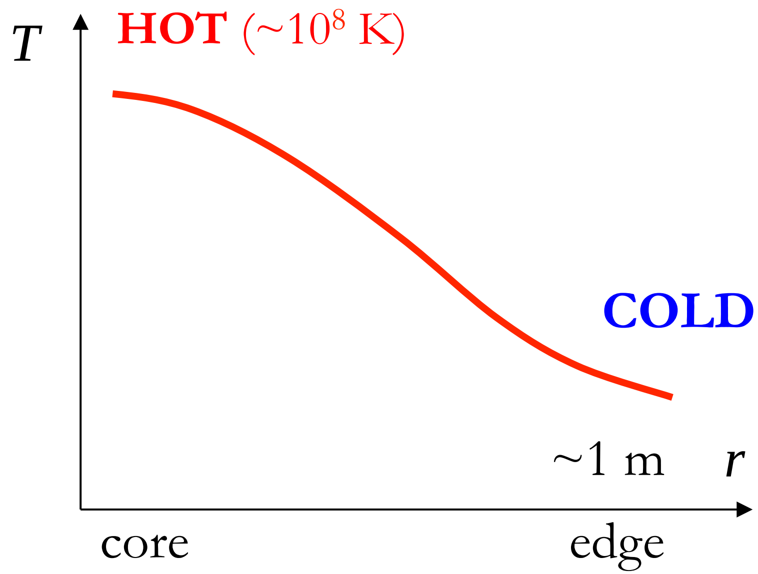

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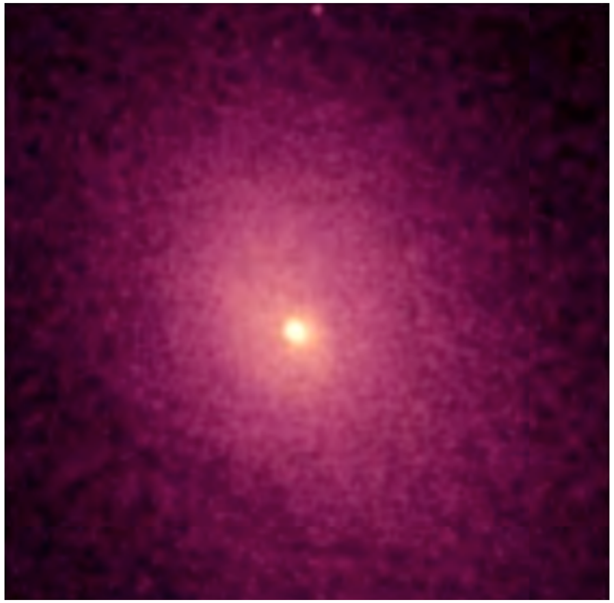
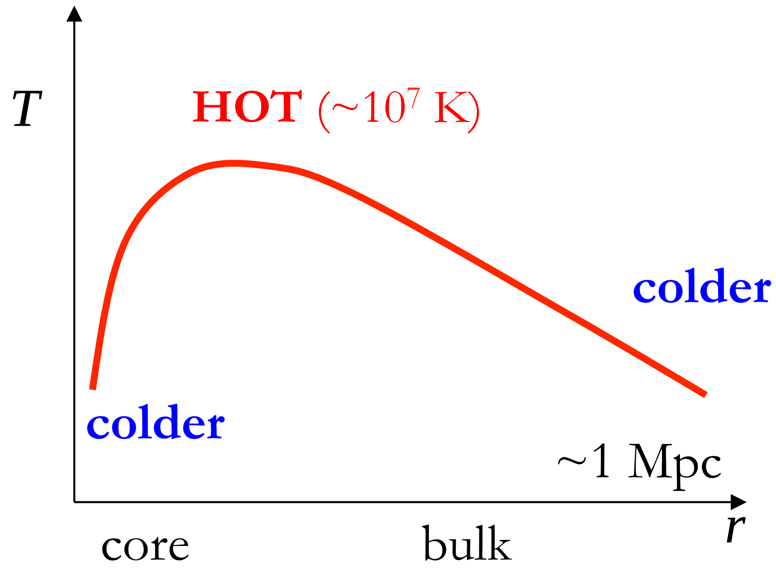


(Abell 2019 in X-ray, Image: Chandra)



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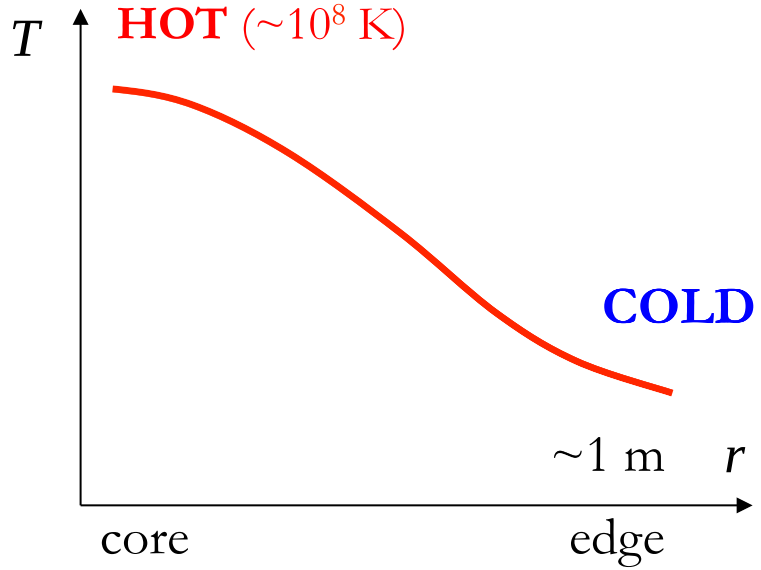

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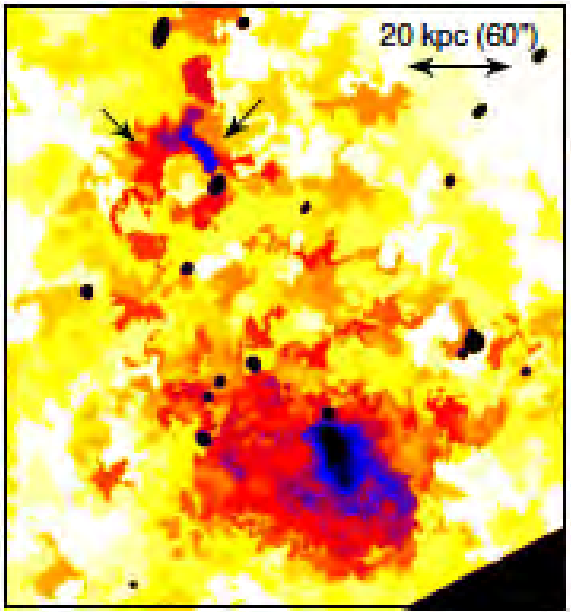
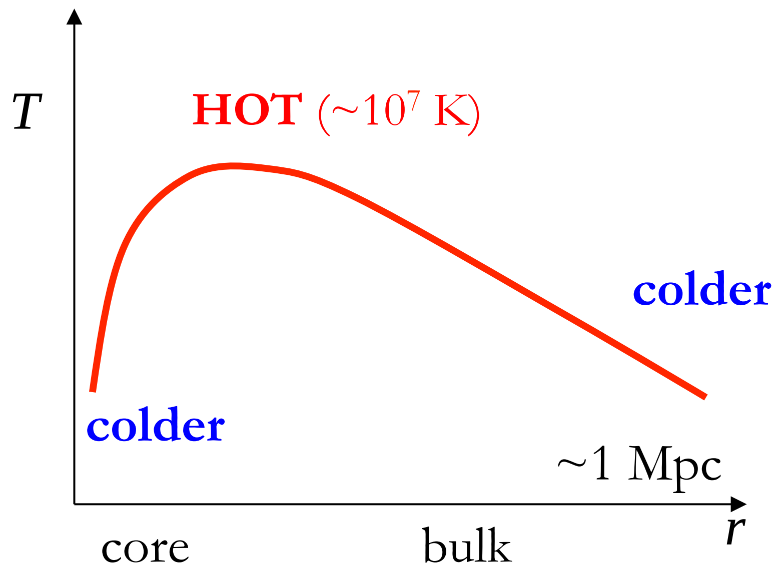


(temperature fluctuation map of Abell 262, J. Sanders et al. 2009)



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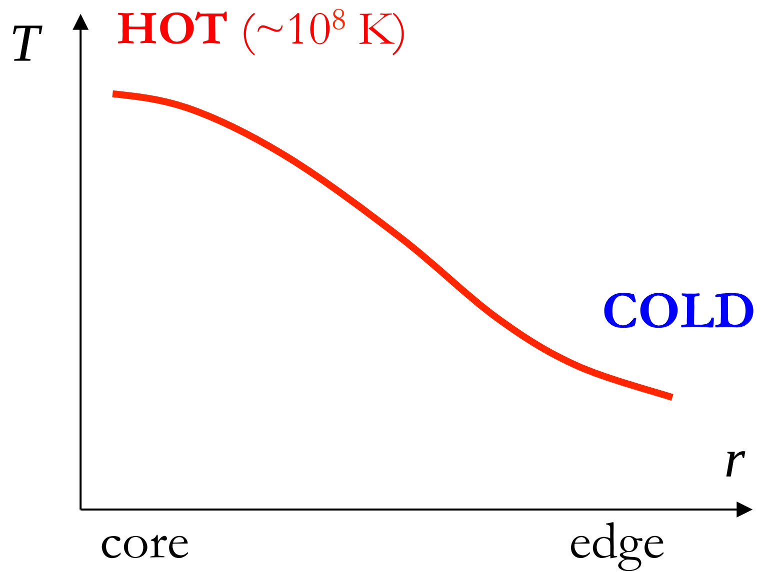

Turbulent transport is much faster than collisional: nature is impatient and will not wait for slow collisions to relax the system to equilibrium!

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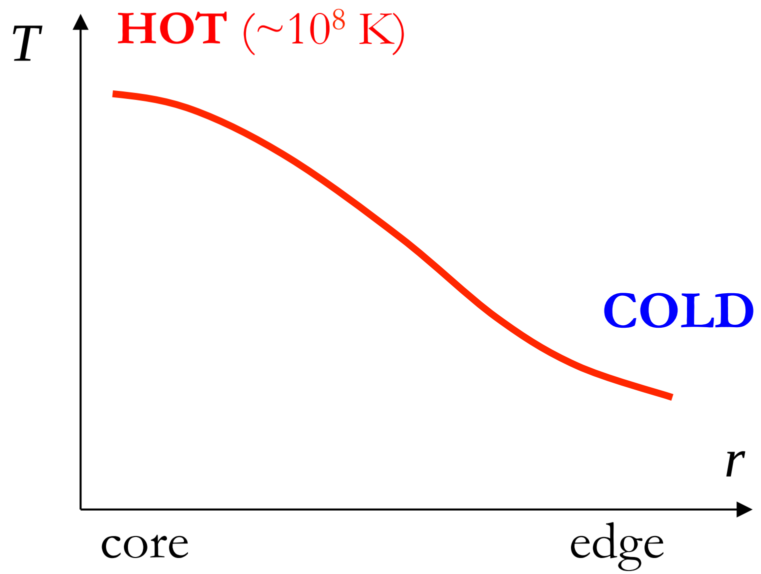

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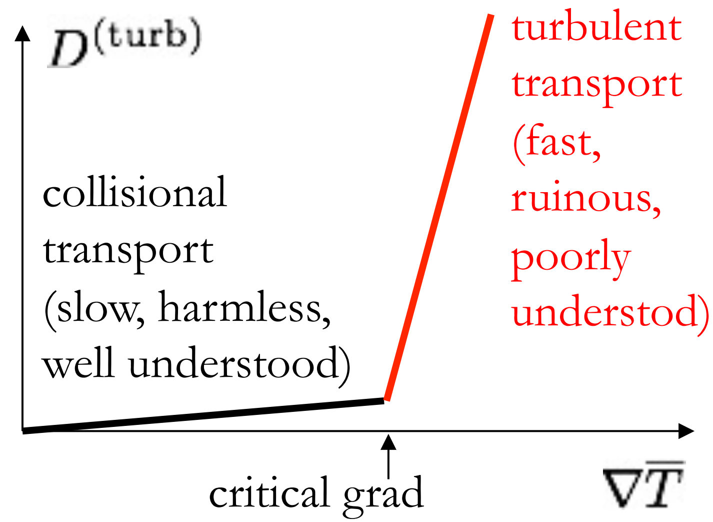
We want to be able to predict $D^{(\mathrm{turb})}$ as a function of everything:
local equilibrium quantities (e.g., $\nabla \overline{T}$),
configuration of the magnetic cage,
energy and momentum inputs…



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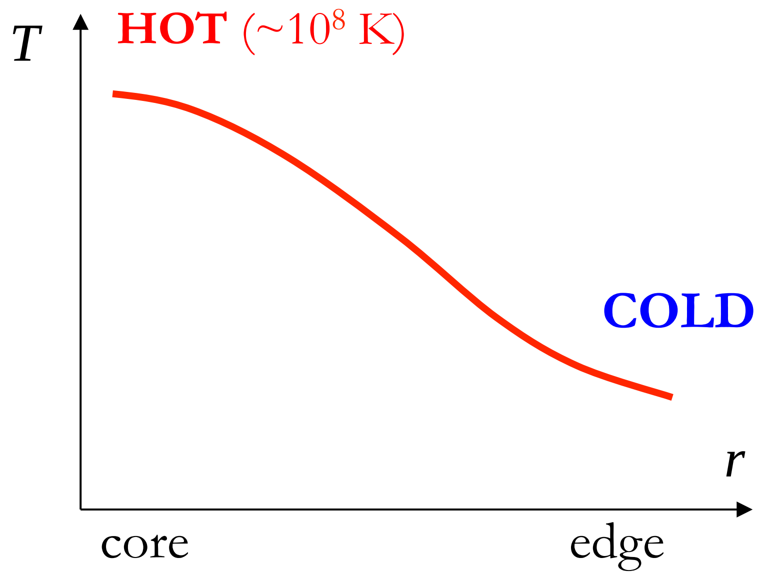

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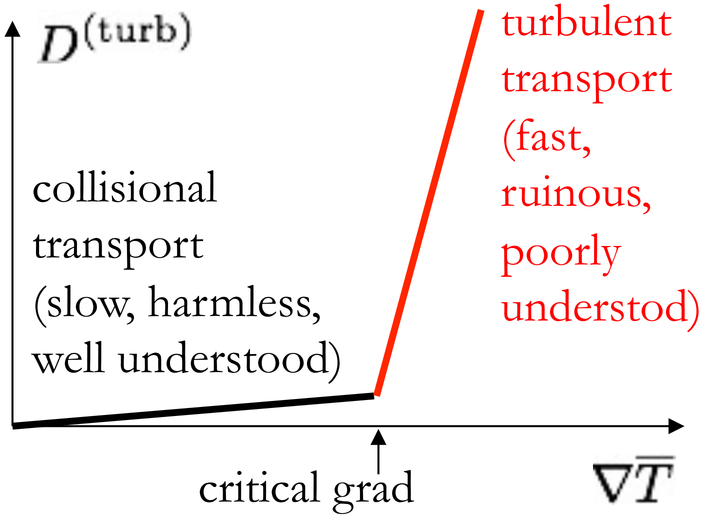
So turbulence is the enemy.
In order to kill it,
we must understand it
(also because it's a challenge and we must meet it to keep our self-respect as a species)



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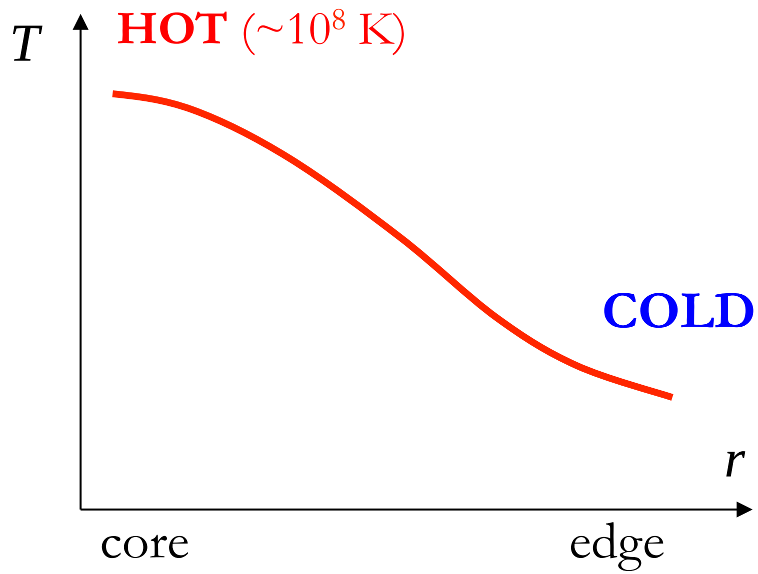

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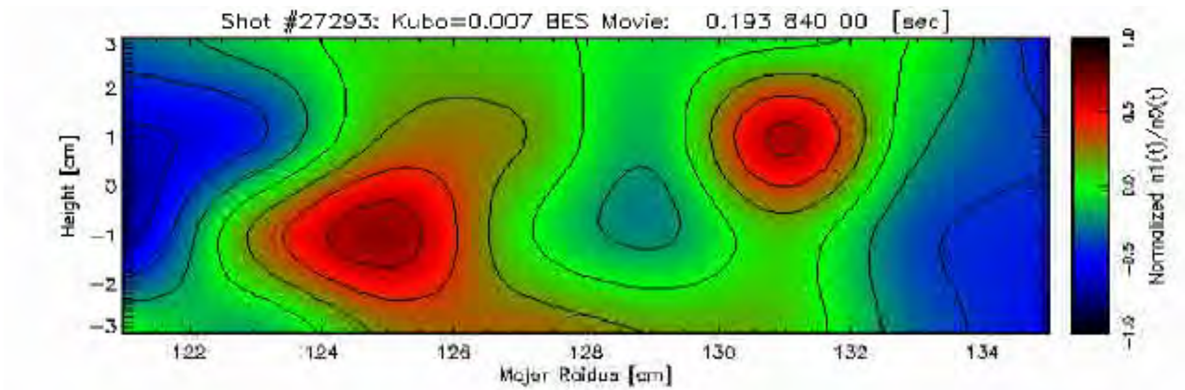

BES image of density fluctuations in MAST
[Movie: Y.-c. Ghim, Oxford]

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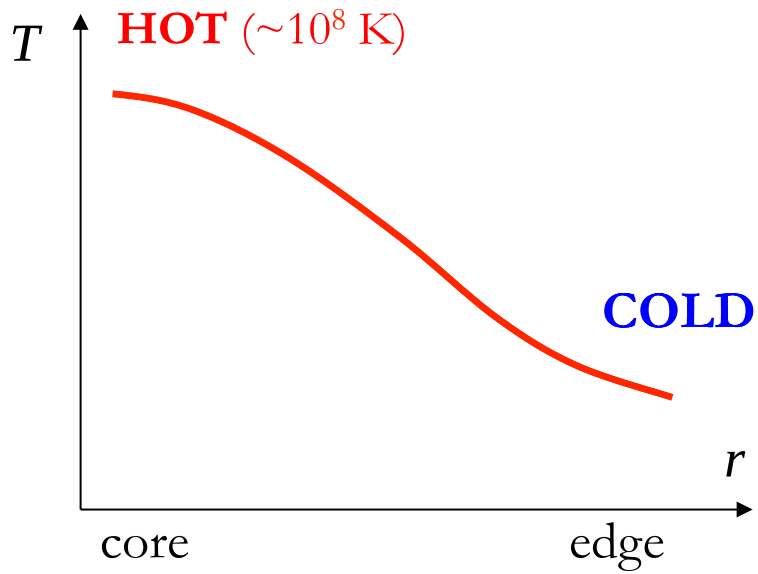

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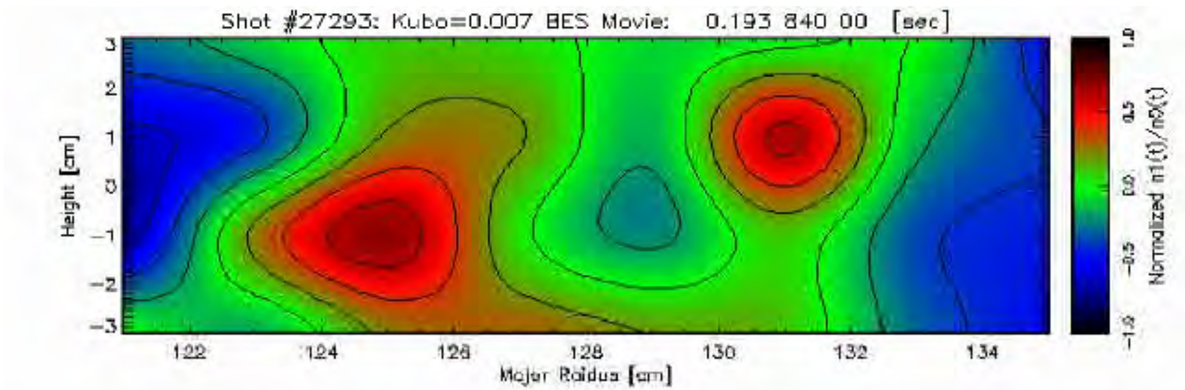
$$D^{(\mathrm{turb})} \sim u^2 \tau_{\mathrm{corr}}$$

$$\tau_{\mathrm{corr}} \sim \frac{\ell}{u}$$

ℓ ← eddy size

u ← eddy turnover velocity

τ_{corr}: eddy turnover time



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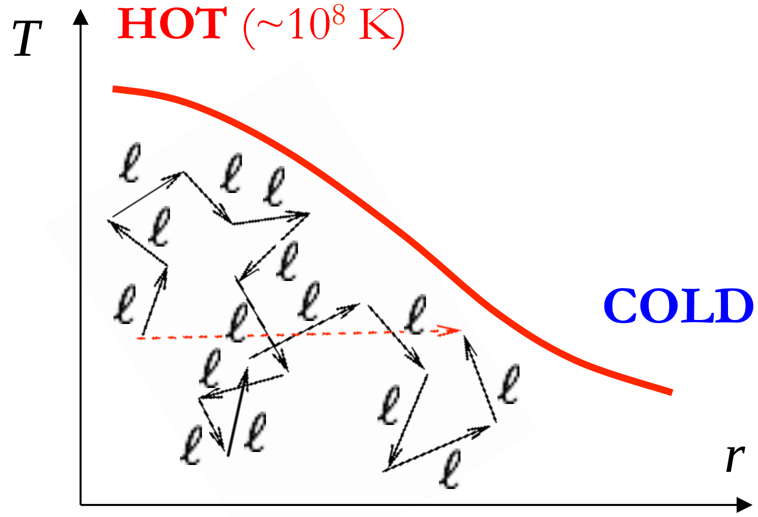


$$D^{(\mathrm{turb})} = \int_0^t dt' \langle u_r(t) u_r(t') \rangle \gg D$$

$$D^{(\mathrm{turb})} \sim u^2 \tau_{\mathrm{corr}} \sim \frac{\ell^2}{\tau_{\mathrm{corr}}} \sim u\ell$$

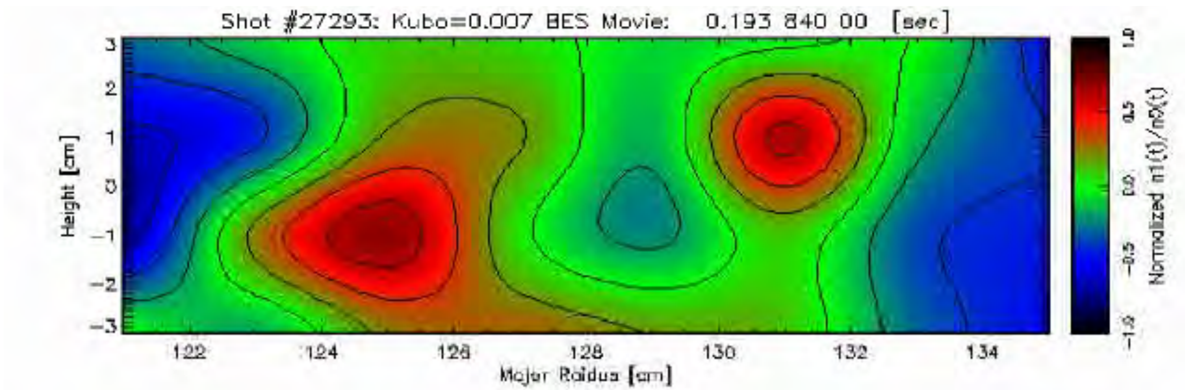
random walk again, but now particles carrying energy hop from eddy to eddy

$$\tau_{\mathrm{corr}} \sim \frac{\ell}{u}$$

ℓ ← eddy size

u ← eddy turnover velocity

τ_{corr}: eddy turnover time



BES image of density fluctuations in MAST
[Movie: Y.-c. Ghim, Oxford]

Turbulent Transport

So the "effective mean field theory" for our system looks like this:

$$\frac{\partial \overline{T}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r \left[D^{(\mathrm{turb})} + D \right] \frac{\partial \overline{T}}{\partial r} + S$$

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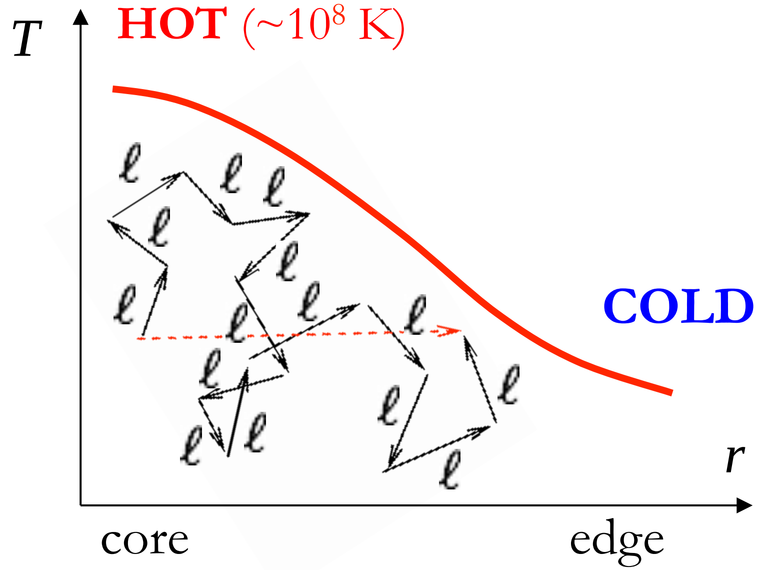
$$\underbrace{\tau_{\mathrm{corr}}}_{\text{eddy turnover time}} \sim \frac{\ell \;\leftarrow \text{eddy size}}{u \;\leftarrow \text{eddy turnover velocity}}$$



random walk again, but now particles carrying energy hop from eddy to eddy

I have started drawing on some notions to do with the nature of turbulence. In the rest of this lecture, I will attempt a very basic and non-rigorous introduction to turbulence…

La turbolenza (how it all started)

Leonardo da Vinci
(1452-1519)

La turbolenza (how it all started)

"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion."

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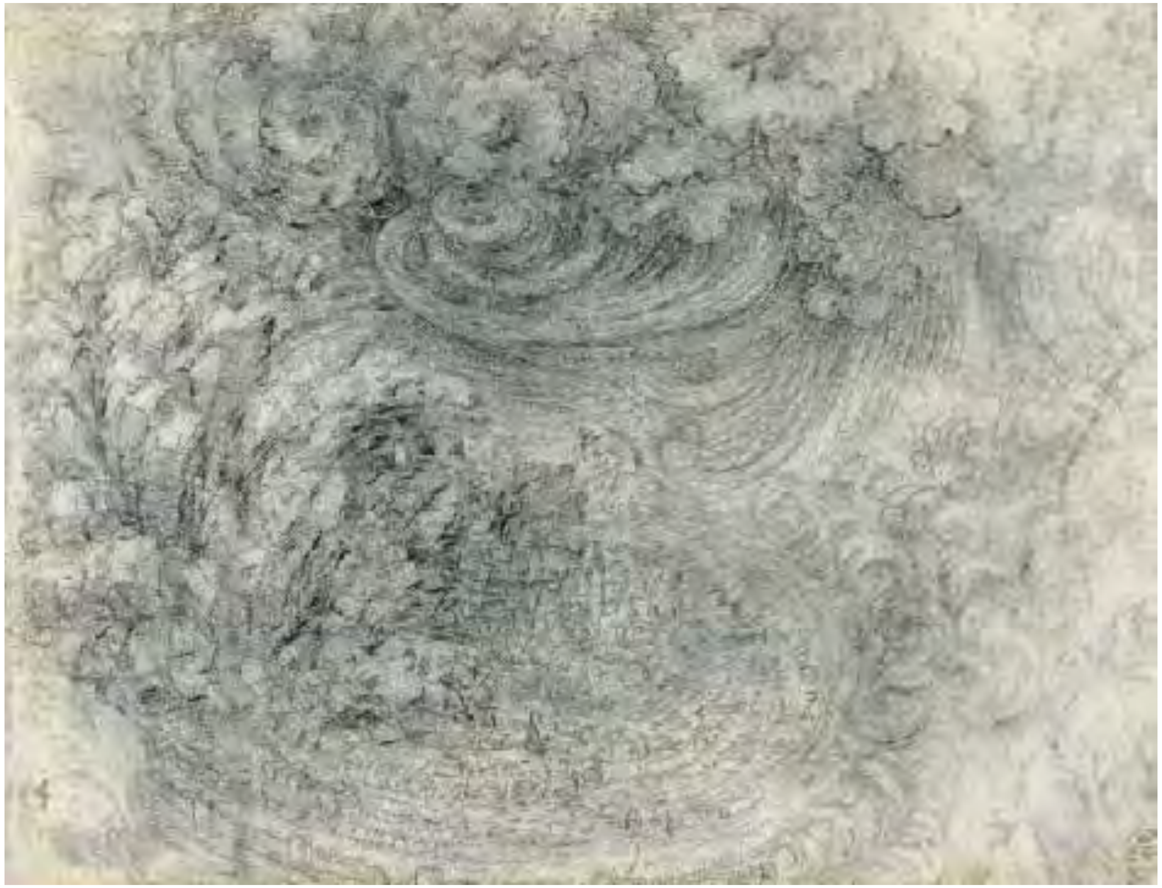

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La turbolenza (how it all started)

So, the basic idea is that a mean, laminar flow breaks up into disordered eddy-like motions

A Universal Phenomenon…

Turbulence in the wake of
Virgin Atlantic Airbus A340 descending to LHR
[Image: Greg Bajor on flickr, 2011]

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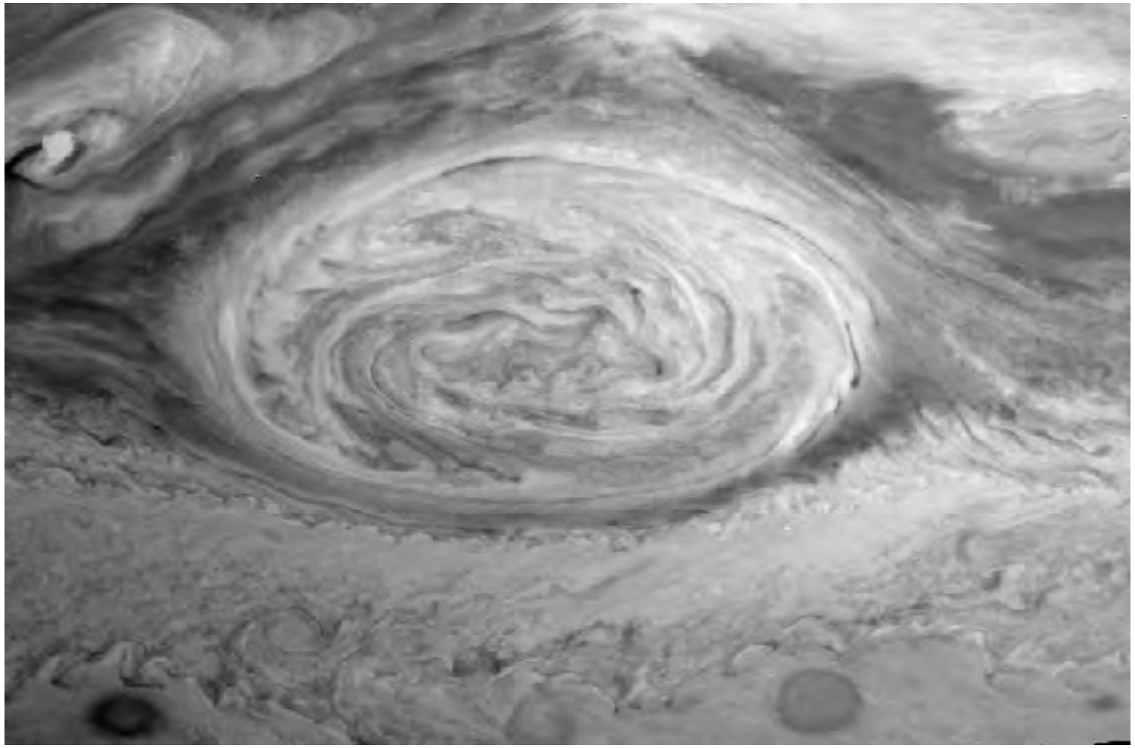
A Universal Phenomenon…



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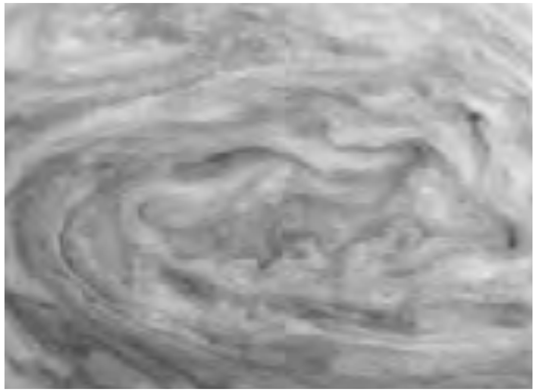
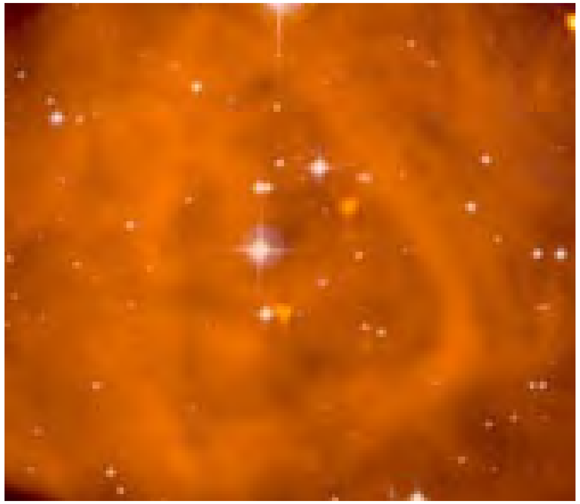


The Great Red Spot of Jupiter
[Image: Galileo, near-infrared (756 nm), 26 June 1996]

A Universal Phenomenon…

Radio Lobes of Fornax A (10^6 light years across)
[Image: Ed Fomalont (NRAO) et al.,
VLA, NRAO, AUI, NSF]

So, the basic idea is that a mean, laminar flow breaks up into disordered eddy-like motions



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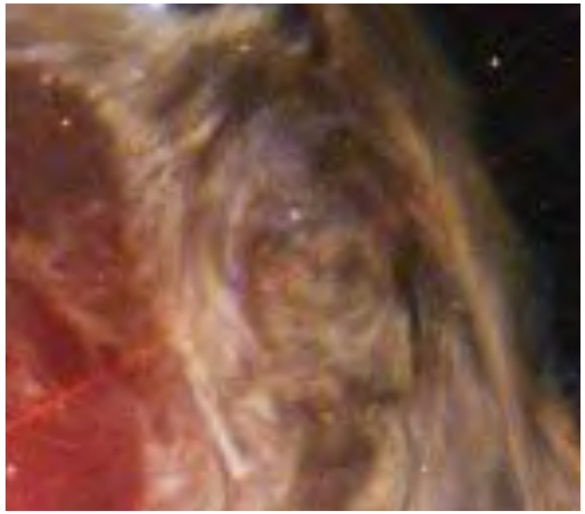

V838 Monocerotis, 20 000 light years away
[Image: Hubble, February 2004]

A Universal Phenomenon…

V. Van Gogh, *The Starry Night*, June 1889
(MoMA, NY)

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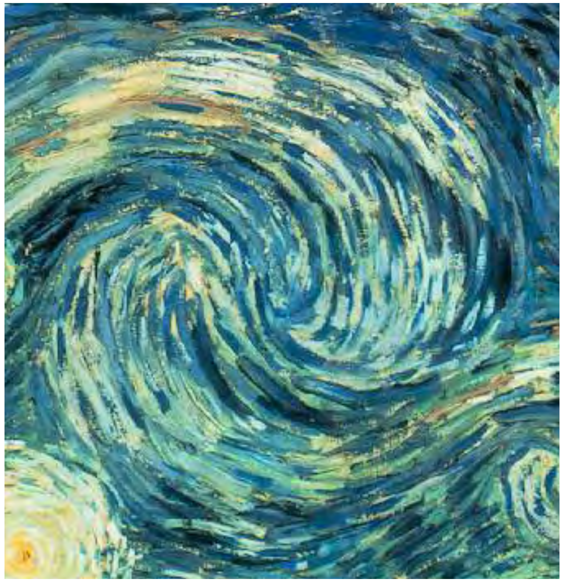
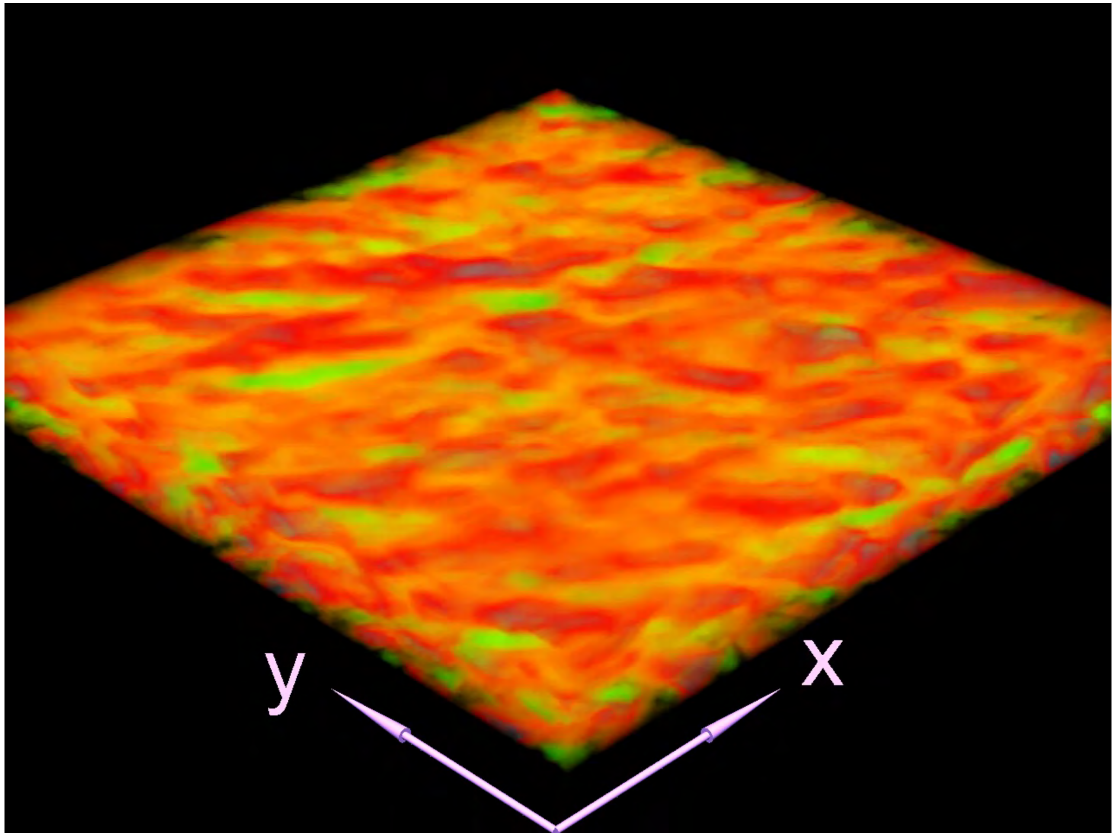


A Universal Phenomenon…



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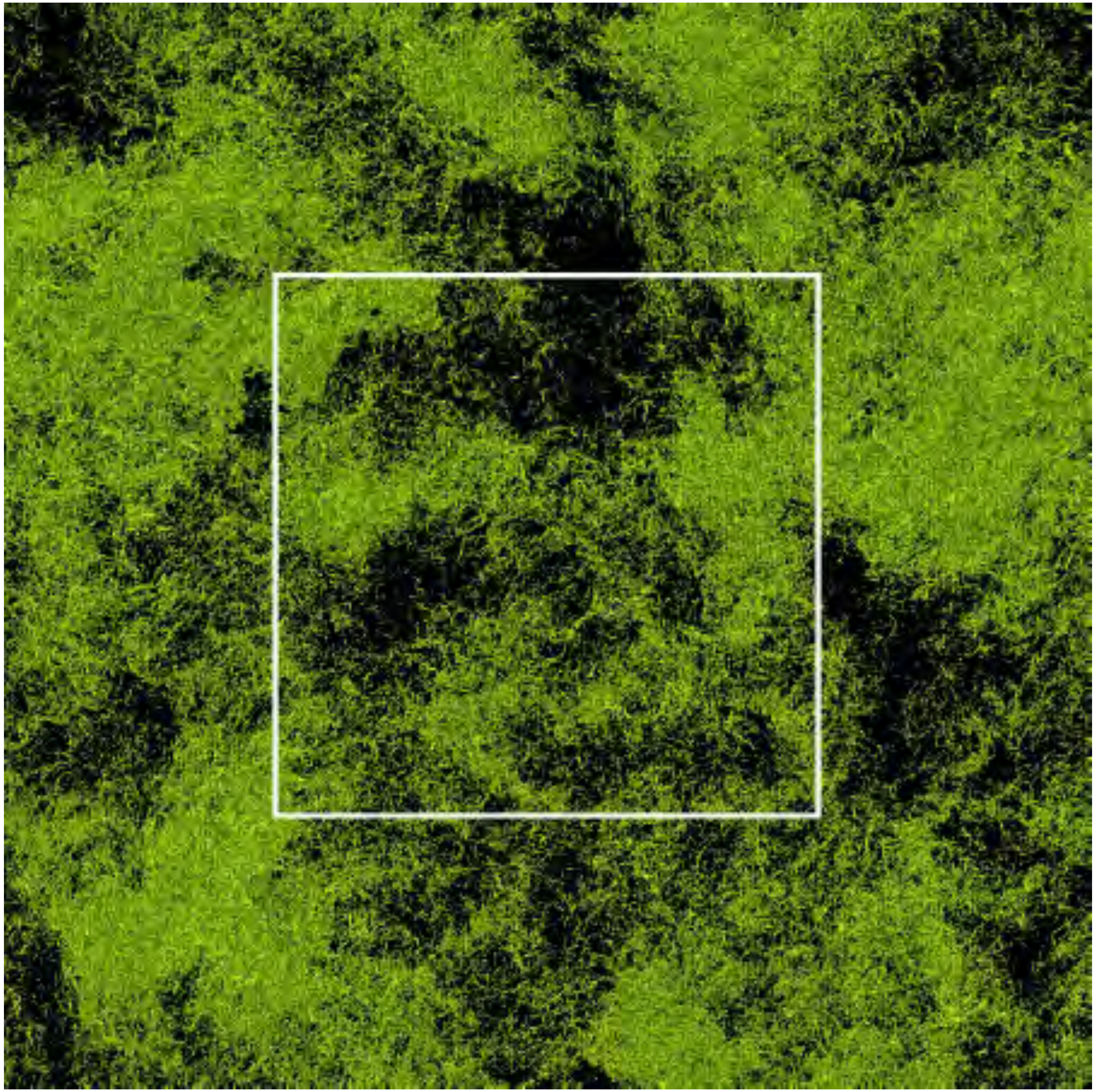
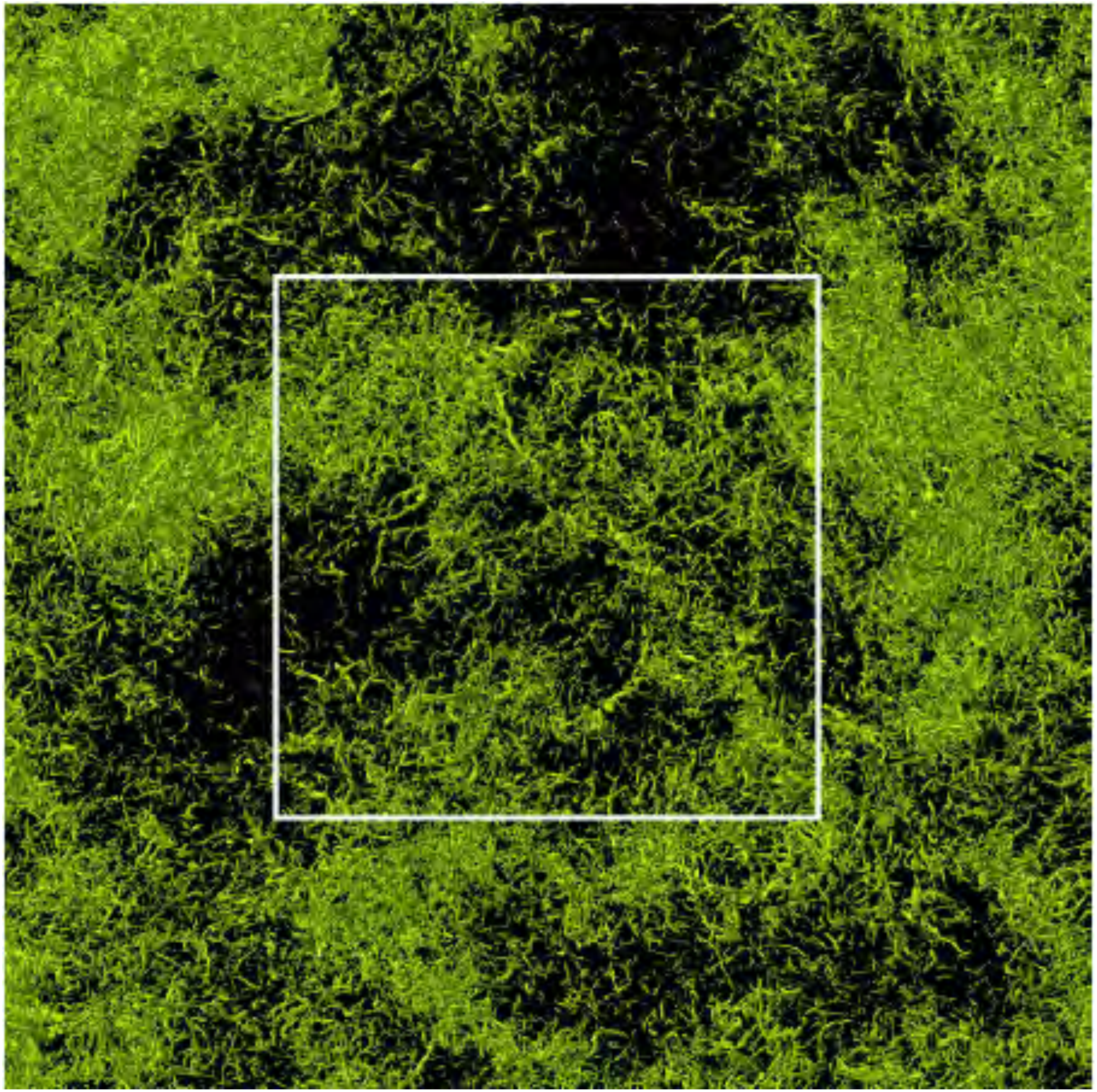
Gyrokinetic simulation of a tokamak
[E. Highcock, Oxford]

What the Structure of These Fluctuations?



[Image: Earth Simulator, 4096^3, isovorticity surfaces; Y. Kaneda]

Turbulence is Multiscale Disorder



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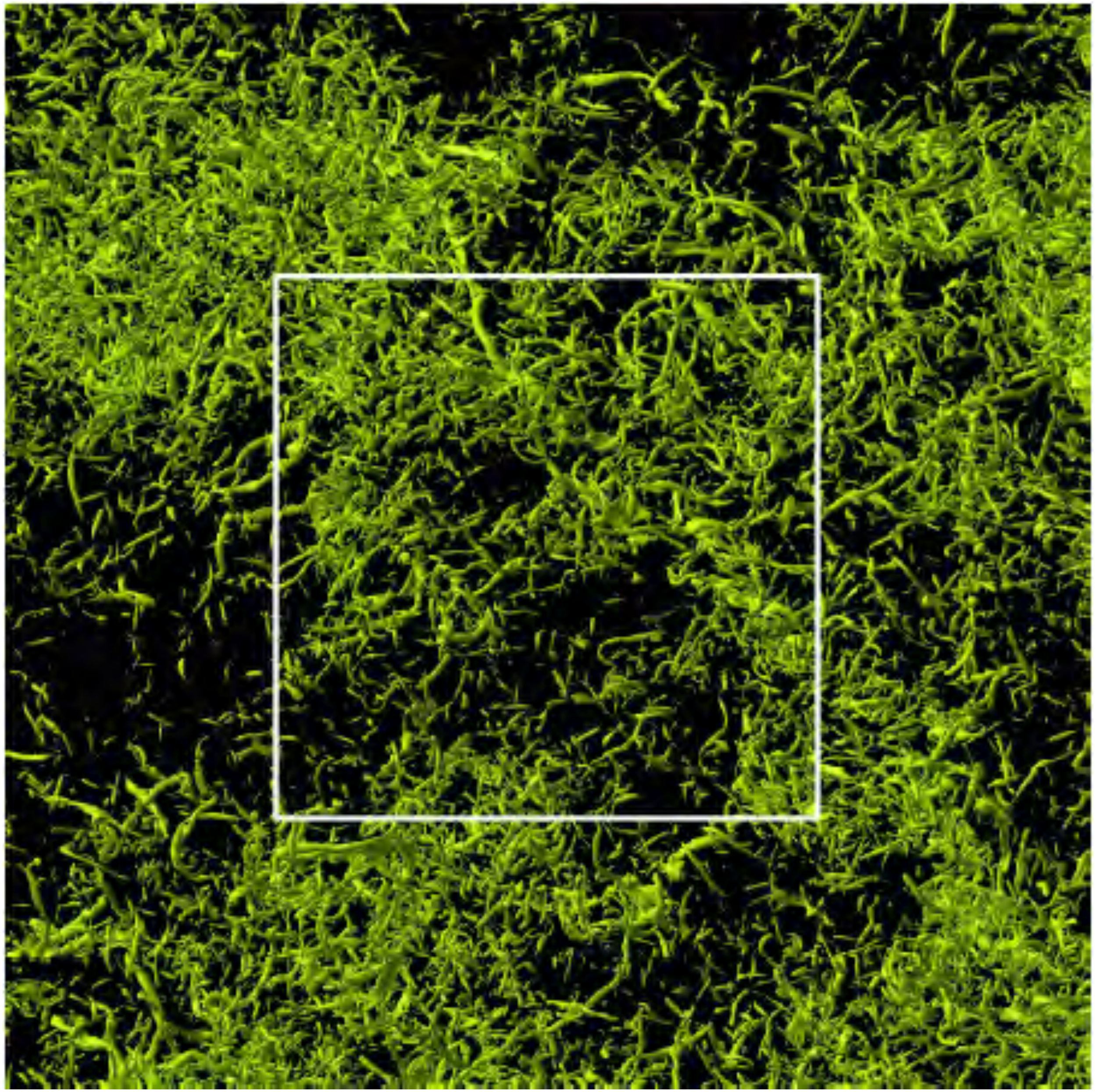

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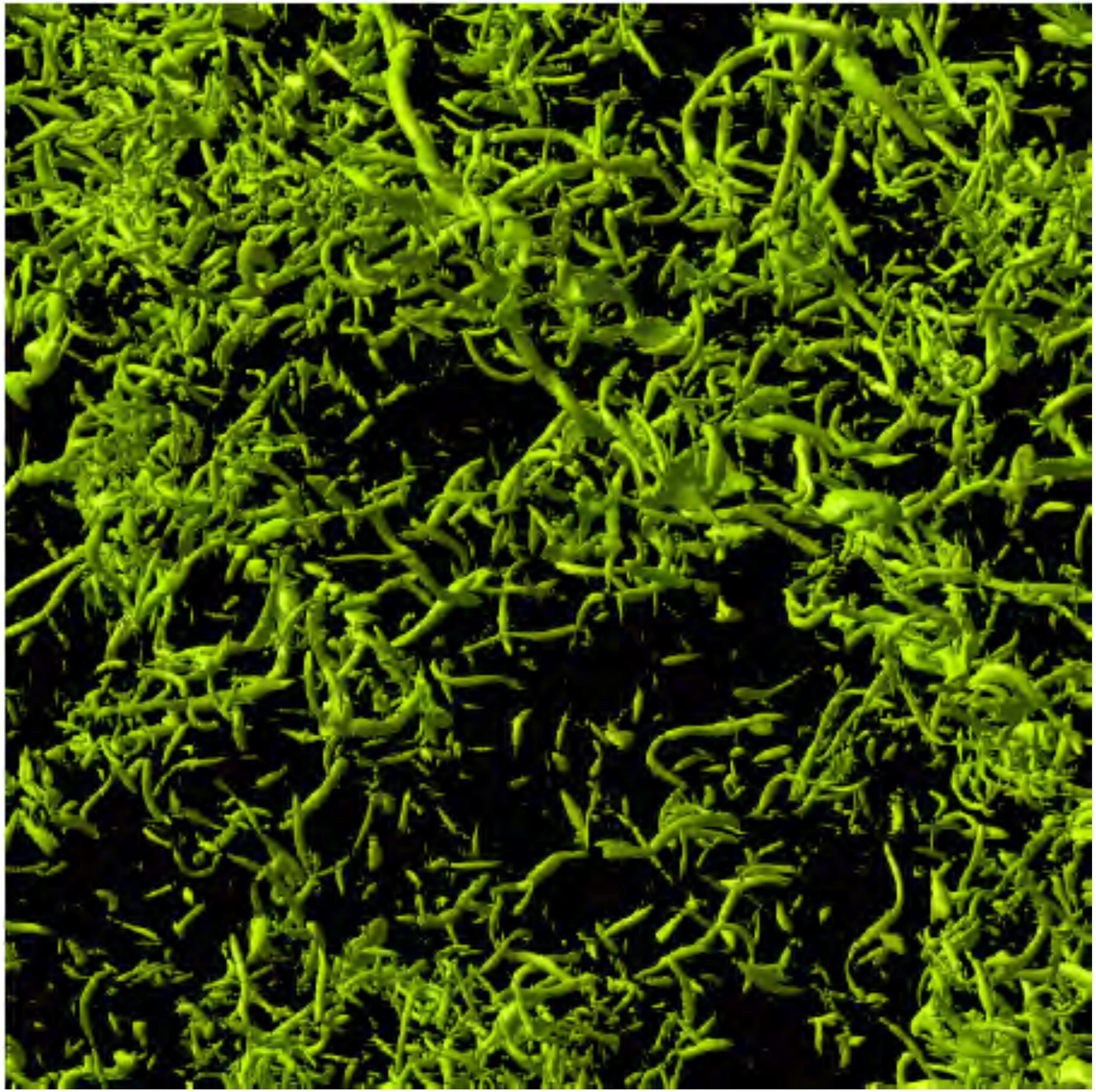

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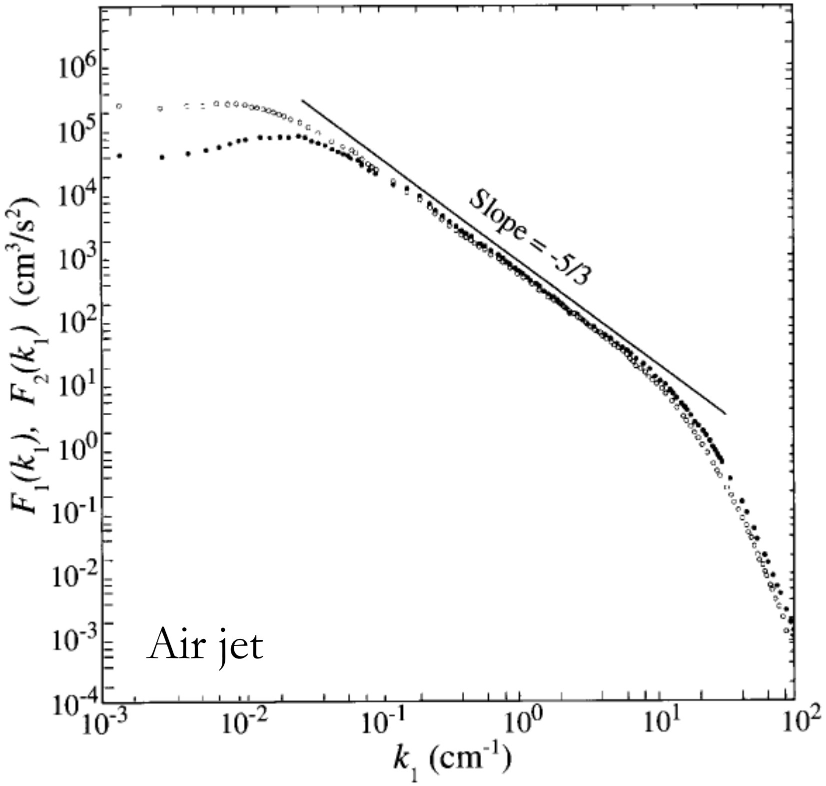
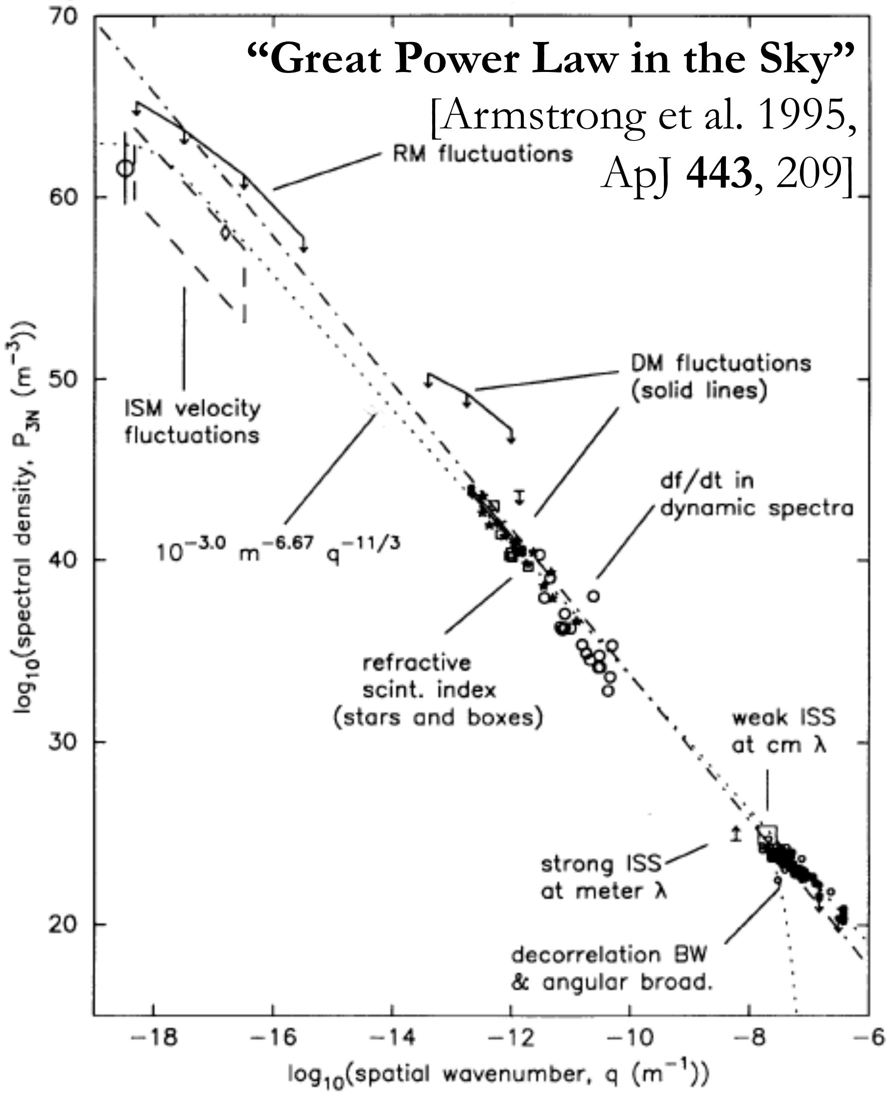
Turbulence is Multiscale Disorder

Gyrokinetic simulation of tokamak turbulence
[R. Waltz & J. Candy, GA, San Diego]

Spectra: Power Laws Galore





"Great Power Law in the Sky"
[Armstrong et al. 1995, ApJ **443**, 209]

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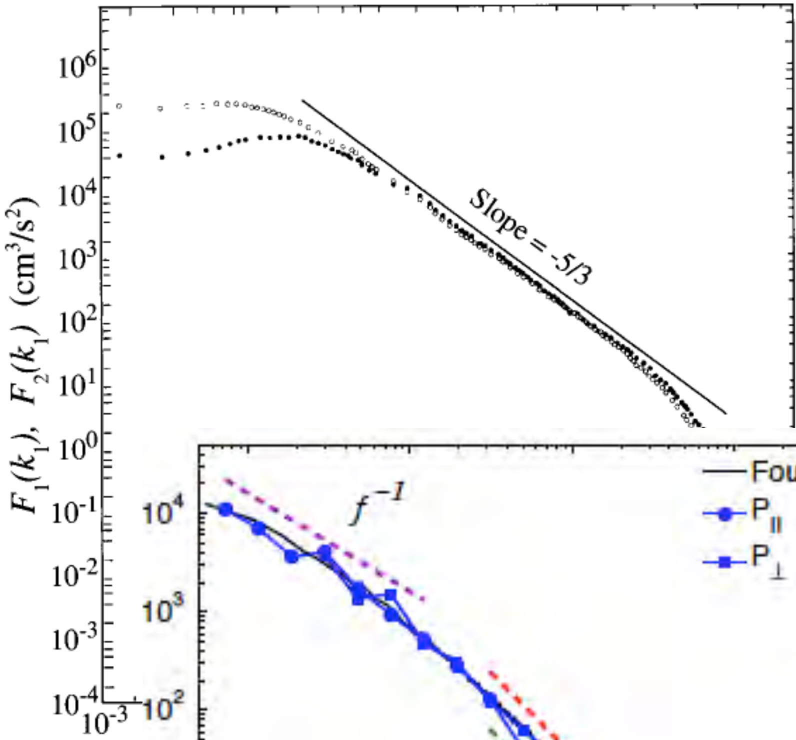
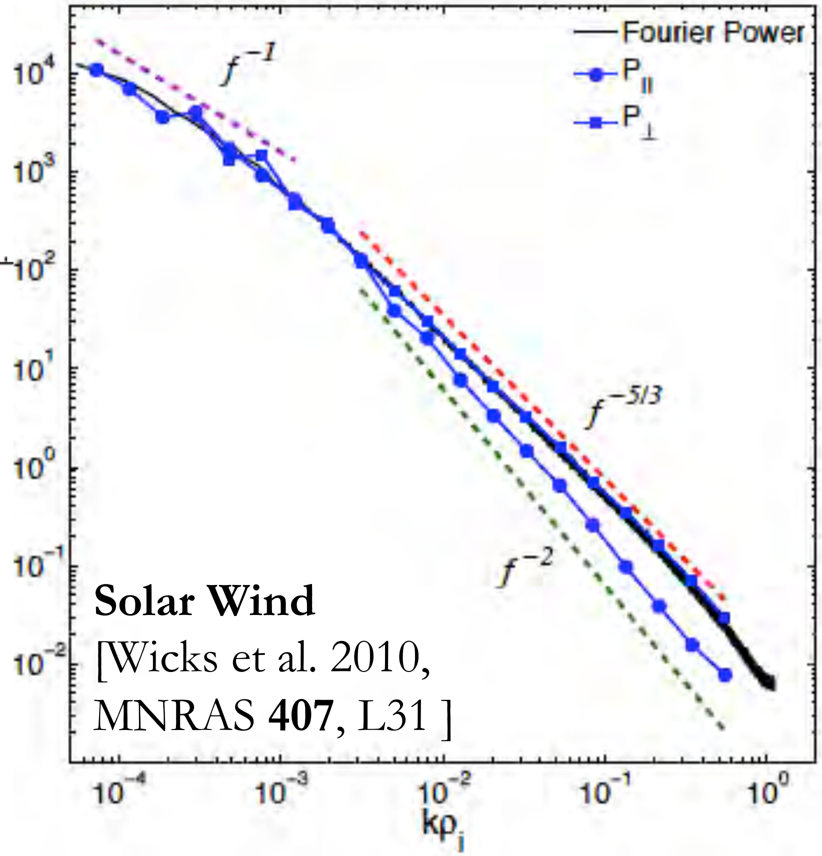
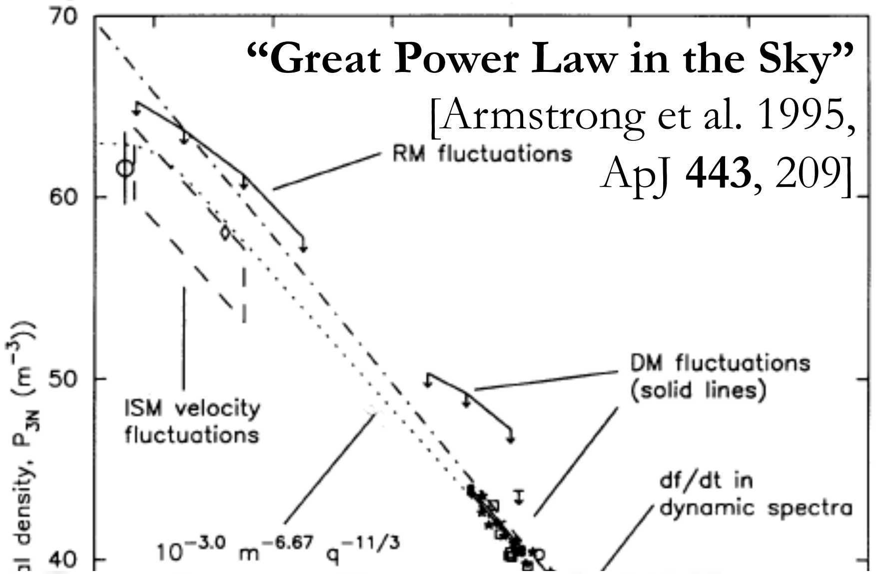




Solar Wind
[Wicks et al. 2010, MNRAS **407**, L31]



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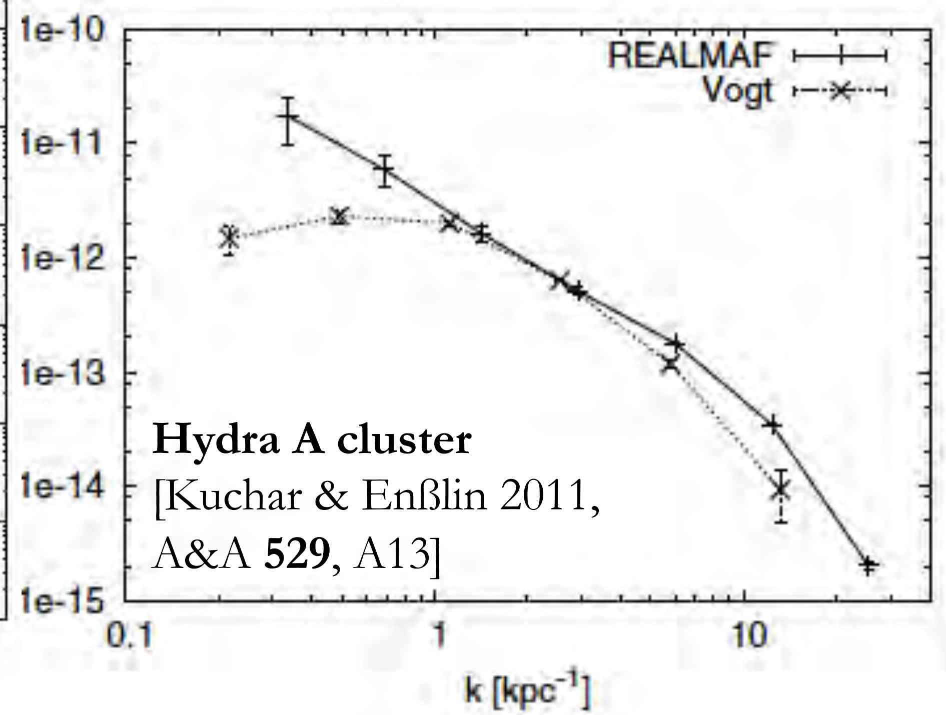
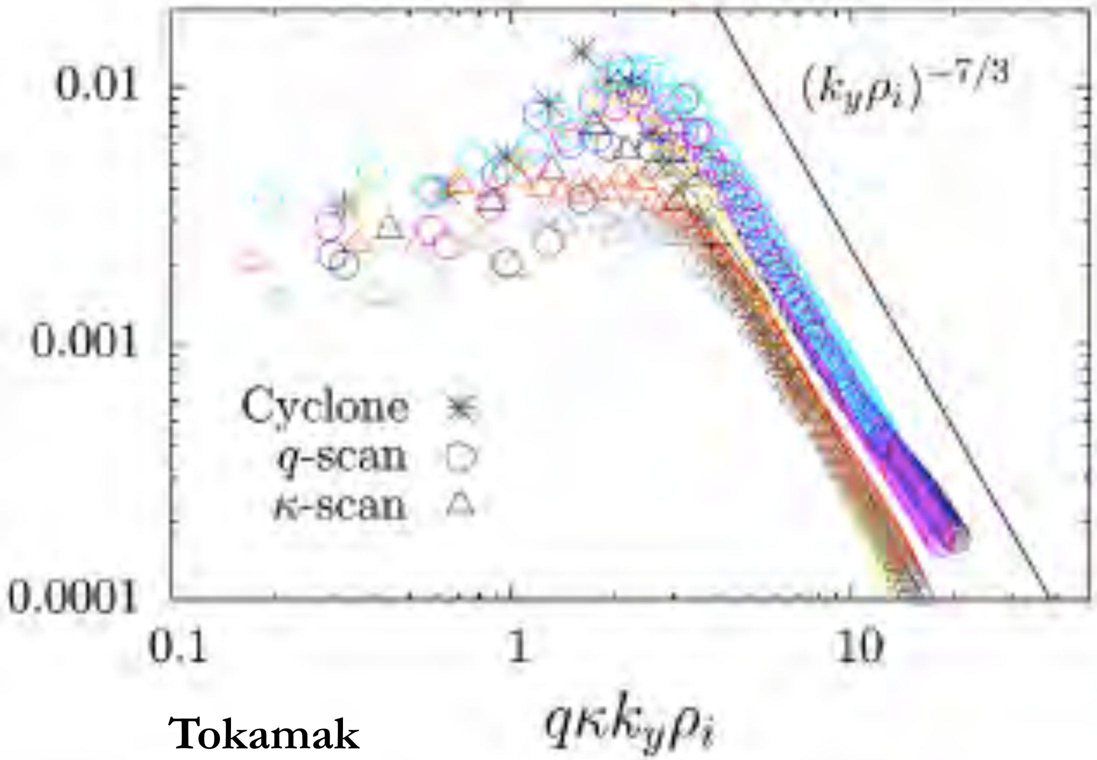


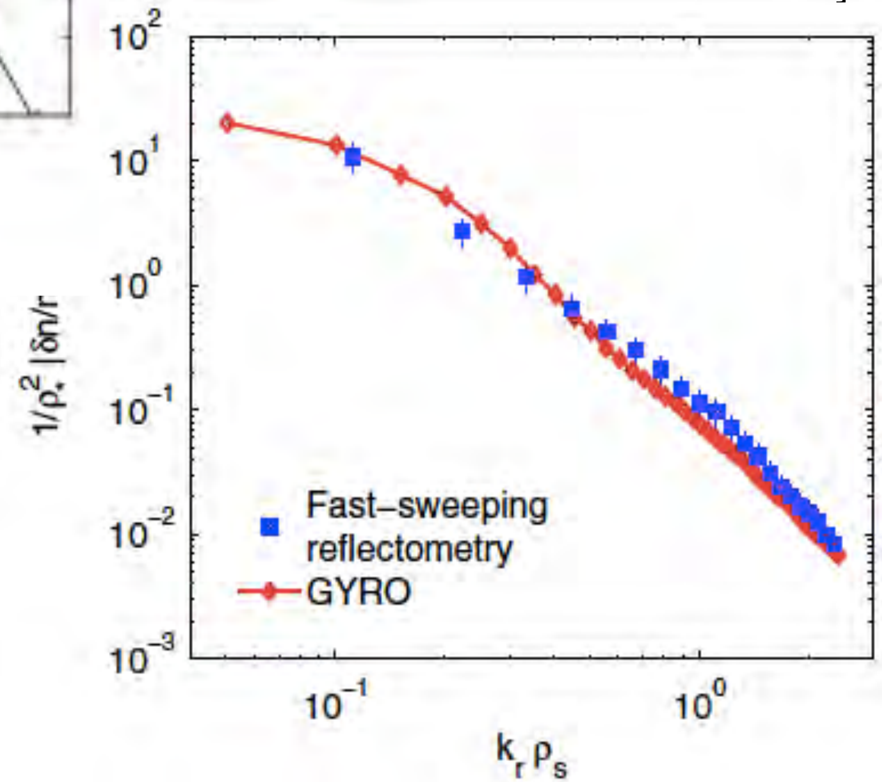
Hydra A cluster
[Kuchar & Enßlin 2011, A&A **529**, A13]

Spectra: Power Laws Galore



Tokamak simulation
[Barnes et al. 2011, PRL **107**, 115003]

TORE SUPRA experiment
[Casati et al. 2011, PRL **102**, 165005]



Why Is Turbulence Multiscale?

Fundamentally, it is about the way in which a nonlinear system processes energy injected into it.

I will provide a simple example of how that works…

Why Is Turbulence Multiscale?

Navier-Stokes Equation: $$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

dissipation (viscosity)

injection (some mechanism for which this is a stand-in)

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injected power

$$\boxed{P_{\text{inj}} = \int\frac{d^3\mathbf{r}}{V}\rho\mathbf{u}\cdot\mathbf{f}}$$

dissipated power

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injected power

dissipated power

$$P_{\rm inj} \sim \frac{\rho u_{\rm rms}^3}{L}$$

depends on outer-scale quantities only

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if estimated at outer scale

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dissipation injection

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But then $$\frac{P_{\rm inj}}{P_{\rm diss}} \sim \frac{u_{\rm rms}L}{\nu} = \mathrm{Re} \gg 1$$ imbalance!

"Reynolds number"

Why Is Turbulence Multiscale?

Navier-Stokes Equation:
$$\underset{}{\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + \underbrace{\nu\nabla^2\mathbf{u}}_{\text{dissipation}} + \underbrace{\mathbf{f}}_{\text{injection}}}$$

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To balance dissipation with power injection, turbulence makes small scales

How small is an easy dimensional guess:

$$\ell_\nu \sim (\rho\nu^3/P_{\rm inj})^{1/4} \sim L\mathrm{Re}^{-3/4}$$

"Kolmogorov scale"

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$$\underset{\text{injection}}{L} \gg \underset{\text{“inertial range”}}{\ell} \gg \underset{\text{dissipation}}{\ell_\nu} \sim (\rho\nu^3/P_{\rm inj})^{1/4} \sim L\mathrm{Re}^{-3/4}$$

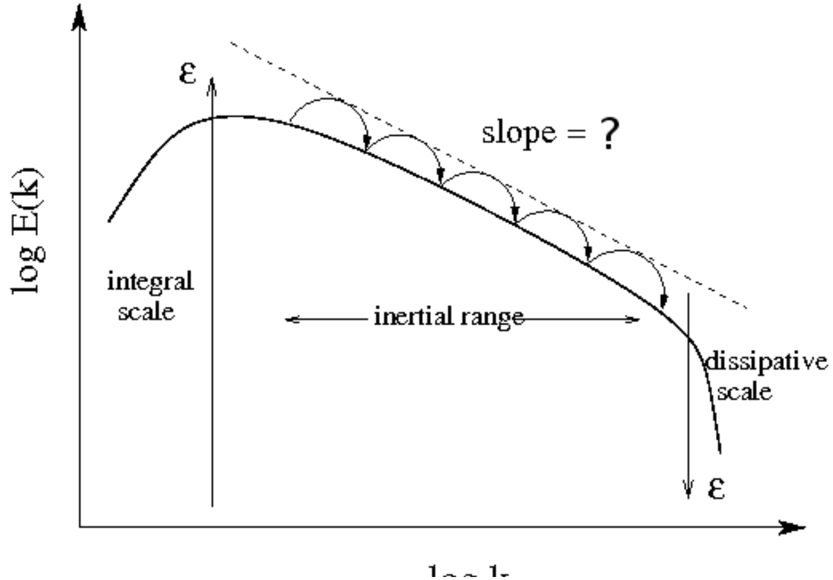
“Kolmogorov scale”

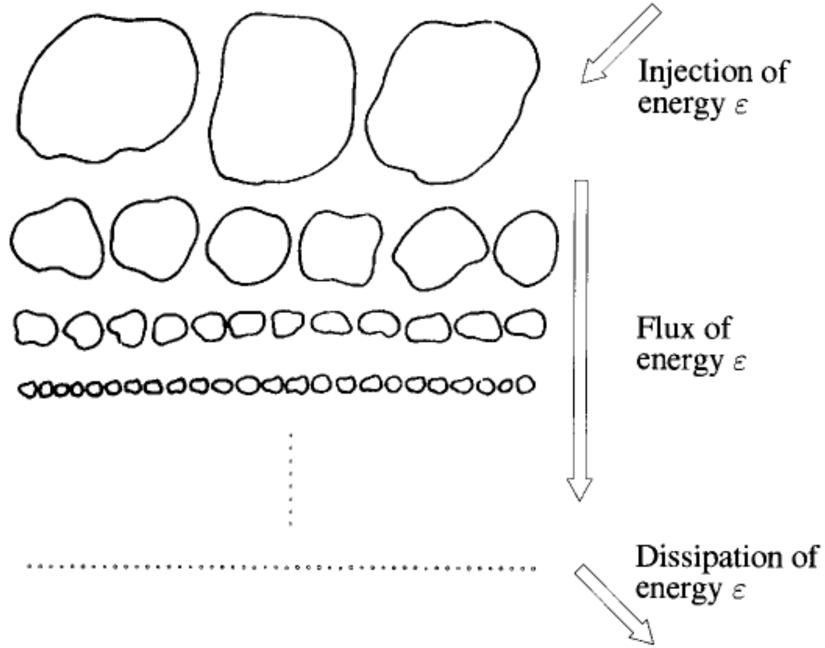
The Richardson Cascade

Lewis Fry Richardson F.R.S.
(1881-1953)

Big whorls have little whorls
  That feed on their velocity,
And little whorls have lesser whorls
  And so on to viscosity.

1922





The Jonathan Swift Cascade

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1922

Jonathan Swift
(1667-1745)

So, nat'ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller yet to bite 'em,
And so proceed ad infinitum.
Thus every poet, in his kind,
Is bit by him that comes behind.

The Kolmogorov Cascade

A. N. Kolmogorov
(1903-1987)

- Universality *(no special systems)*
- Homogeneity *(no special locations)*
- Isotropy *(no special directions)*
- Locality *(no special scales)*

Any broken symmetries are restored in the inertial range…

We wish to predict $\delta u(\ell) = u(r+\ell) - u(r)$

At each scale, $\dfrac{\rho\, \delta u(\ell)^2}{\underbrace{\tau(\ell)}_{\text{“cascade time”}}} \sim P_{\mathrm{inj}} = \mathrm{const}$

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"cascade time"

Dimensionally, $\tau(\ell) \sim \ell/\delta u(\ell)$

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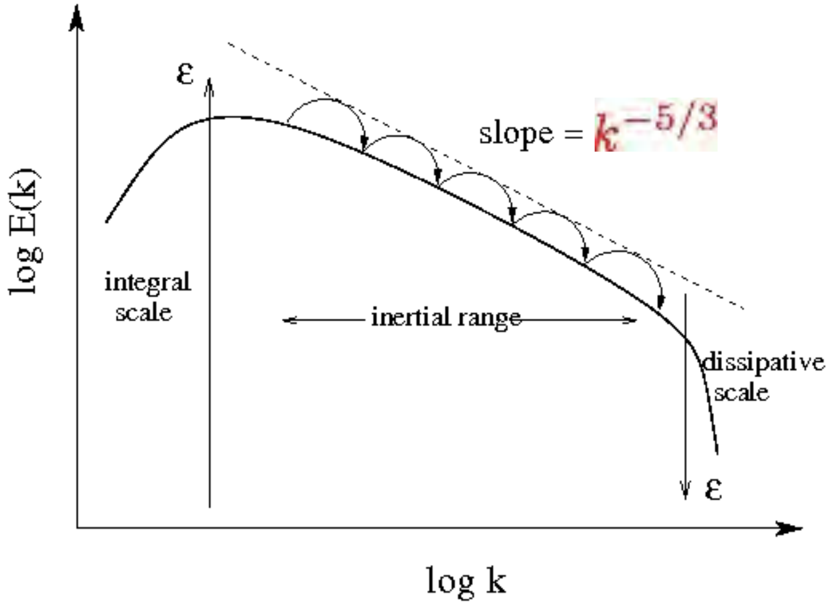

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$$\delta u(\ell)^2 \sim \int_{1/\ell}^{\infty} dk E(k) \quad \Rightarrow \quad E(k) \propto k^{-5/3}$$

"Kolmogorov spectrum"

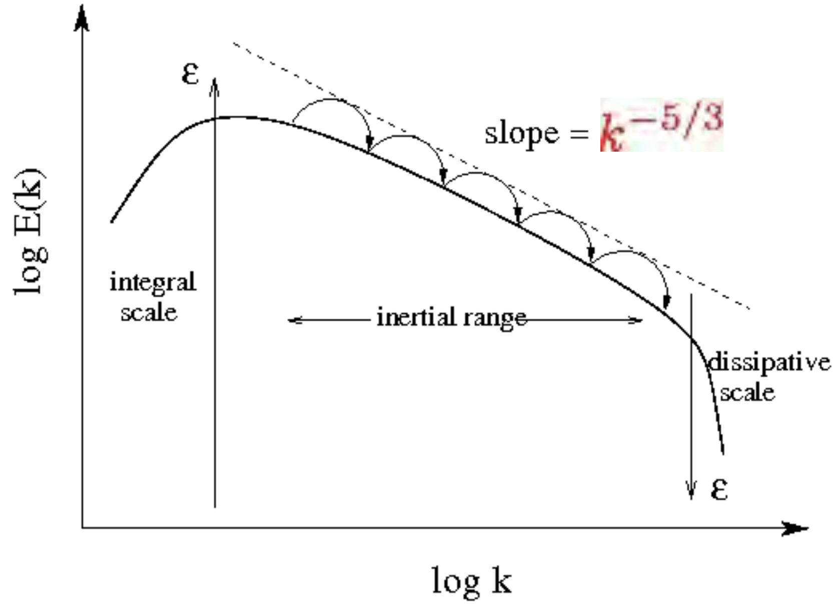
Back to Turbulent Transport…

Recall that
turbulent diffusivity is

$$D^{(\mathrm{turb})} \sim \delta u(\ell)\ell$$
$$\propto \ell^{4/3}$$

Thus, the largest-scale eddies make the largest contribution
to the turbulent transport

The interesting practical question is what that scale is and how fast these eddies are



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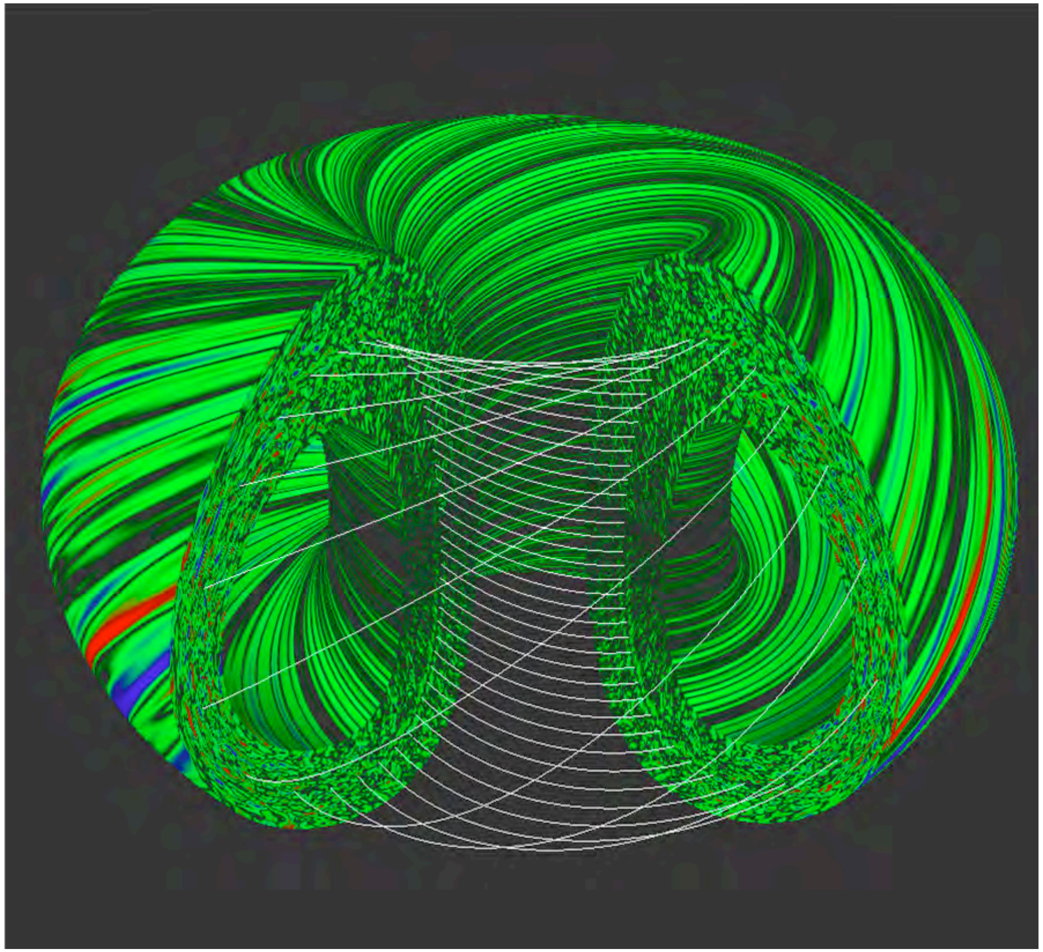
Further Complications…

1. Turbulence in a tokamak is not homogeneous: conditions vary with radius, so we theorise/simulate locally on magnetic surfaces;



[Image: W. Dorland]

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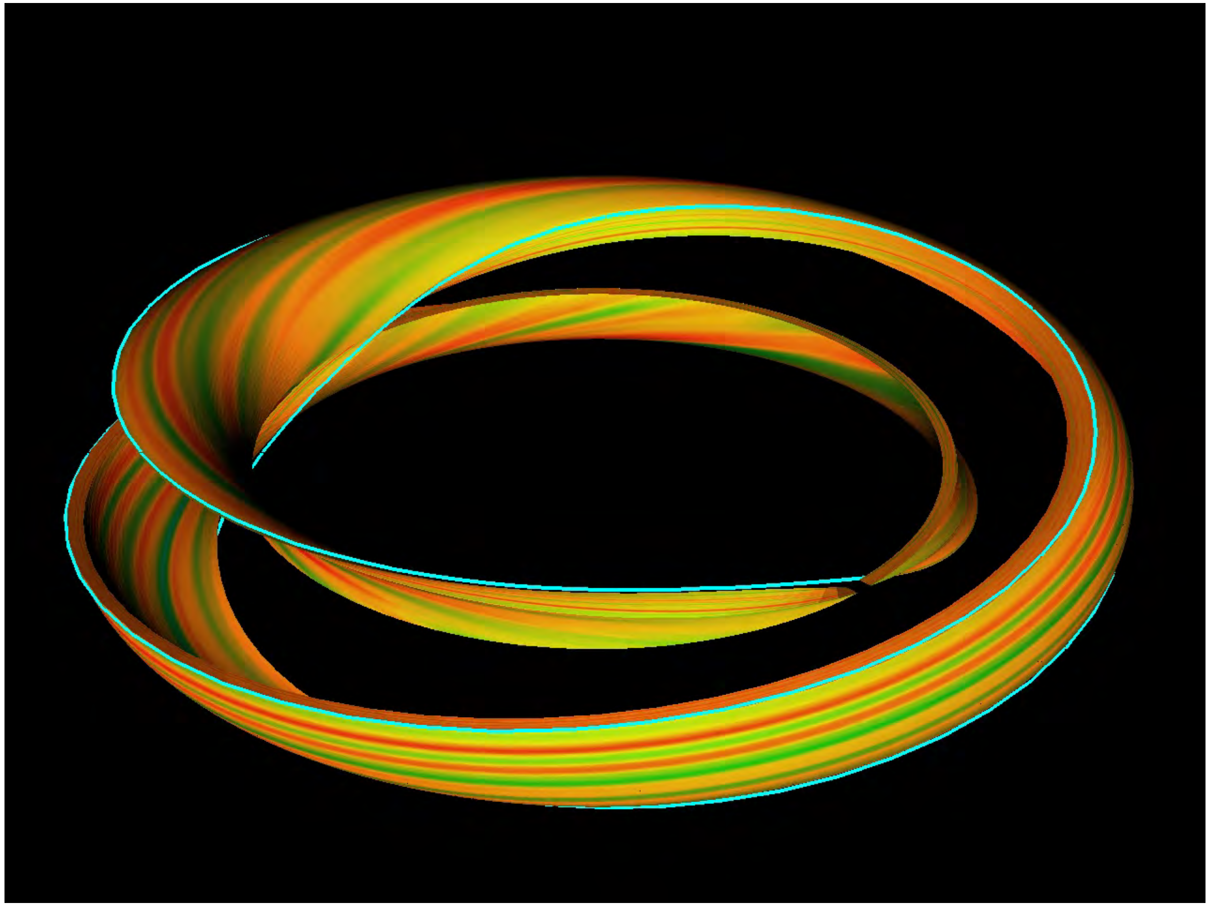

[Illustration:
E. Highcock, Oxford]

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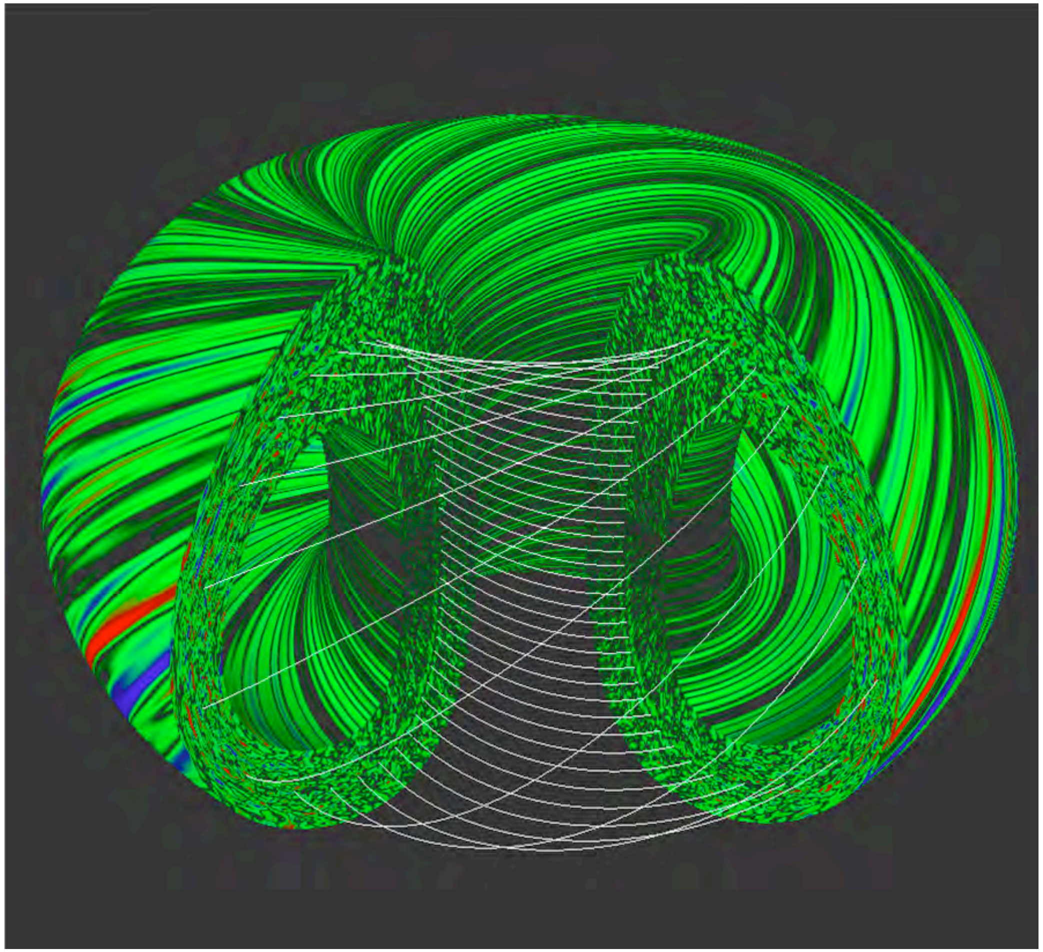
2. Turbulence in a tokamak is not isotropic:
everything is highly stretched along the magnetic field;
this requires some new theoretical concepts concerning the interplay of nonlinear energy cascade and linear wave propagation (along the magnetic field)



[Image: W. Dorland]

Further Complications…

3. Turbulence in a tokamak (and generally in plasmas) is not in a 3D space: in reality the plasma is described by a kinetic equation for the particle distribution function (PDF),

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = C[f]$$

particle streaming　　electric field　　Lorentz force　　collisions

The PDF $f(t,\mathbf{r},\mathbf{v})$ is a field in a 6D phase space. In a turbulent system, small scales will develop not just in **r** but also in **v** (the $\mathbf{v}\cdot\nabla f$ term is a shear in phase space, leading to "phase mixing," i.e., formation of large gradients in velocity space). Thus we have to understand the cascade of energy (or, as it in fact turns out, entropy) in a 6D phase space.

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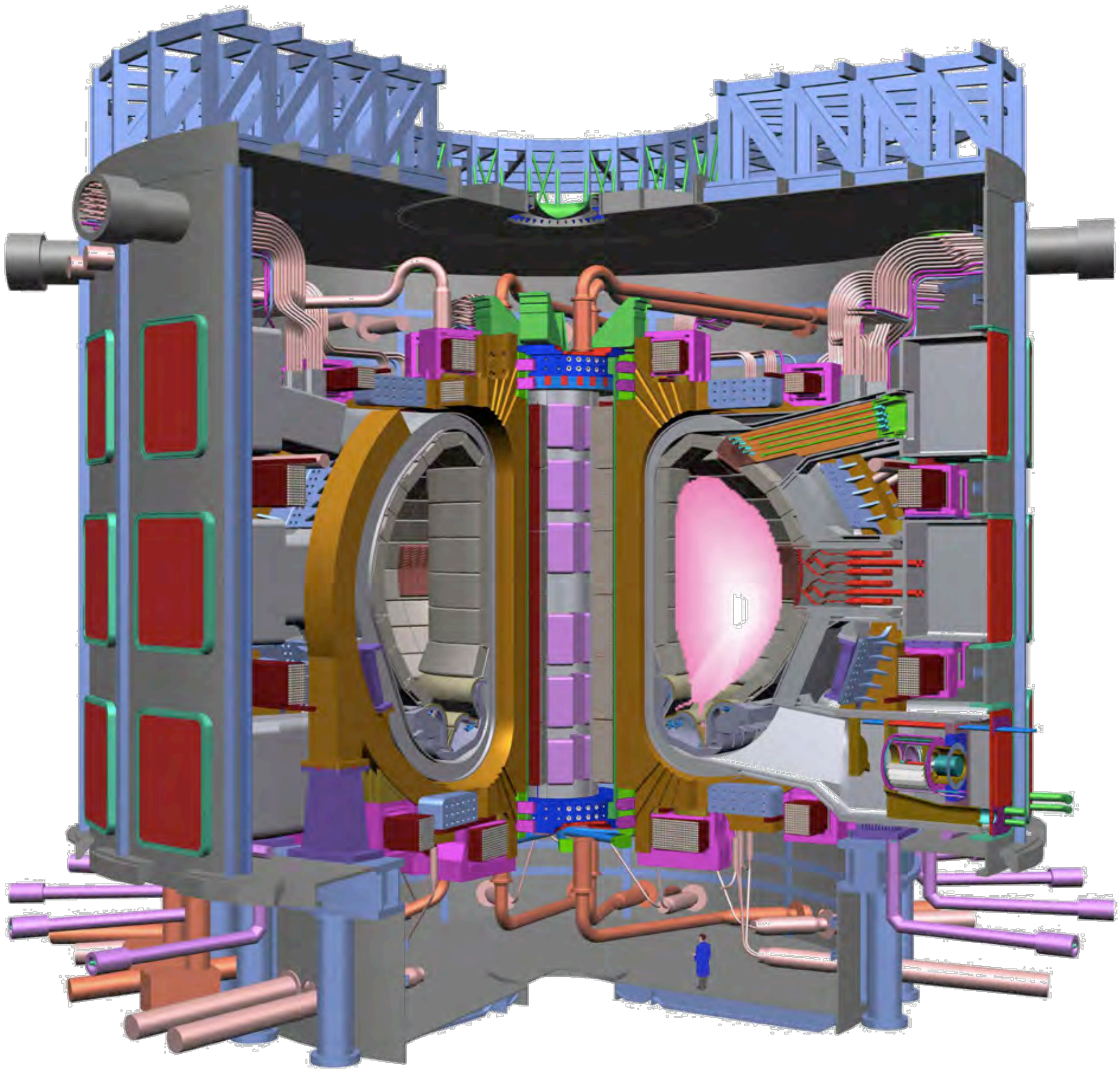
4. You don't want to know what #4 is…

The Story So Far…

- We want to build a machine to tap the energy that fuels stars…



[Image: ITER]

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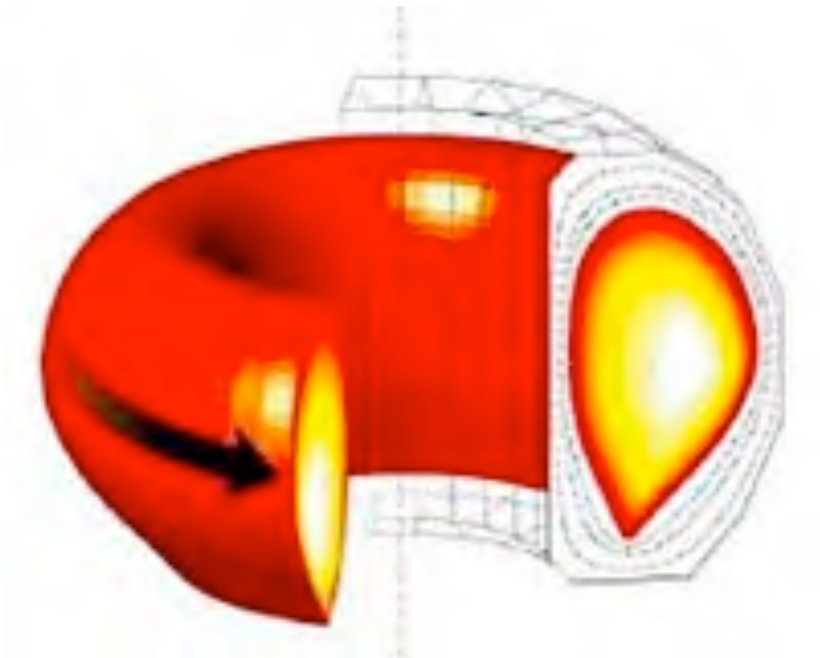
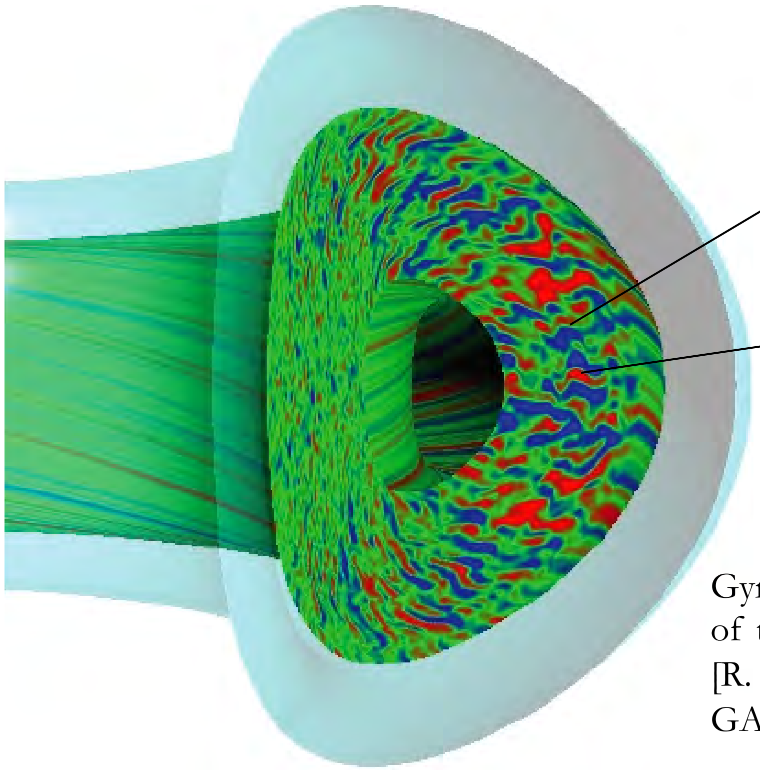
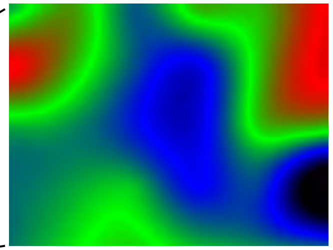


The Story So Far…

- We want to build a machine to tap the energy that fuels stars…
- Inside the machine, plasma is locked in a magnetic cage and kept out of equilibrium (hot inside, cold outside)…
- It rattles its cage, breaks into whirls and swirls in its quest to regain equilibrium… **To keep it in and keep it hot, we must tame the nonlinear beast: turbulence…**



Gyrokinetic simulation
of the DIIID tokamak
[R. Waltz & J. Candy,
GA, San Diego]



BES measurement
of density fluctuations
in the DIIID tokamak
[G. McKee, GA & UW Madison]