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Why is Quantum Gravity so hard?

John Wheater

Morning of Theoretical Physics

January 9th 2021

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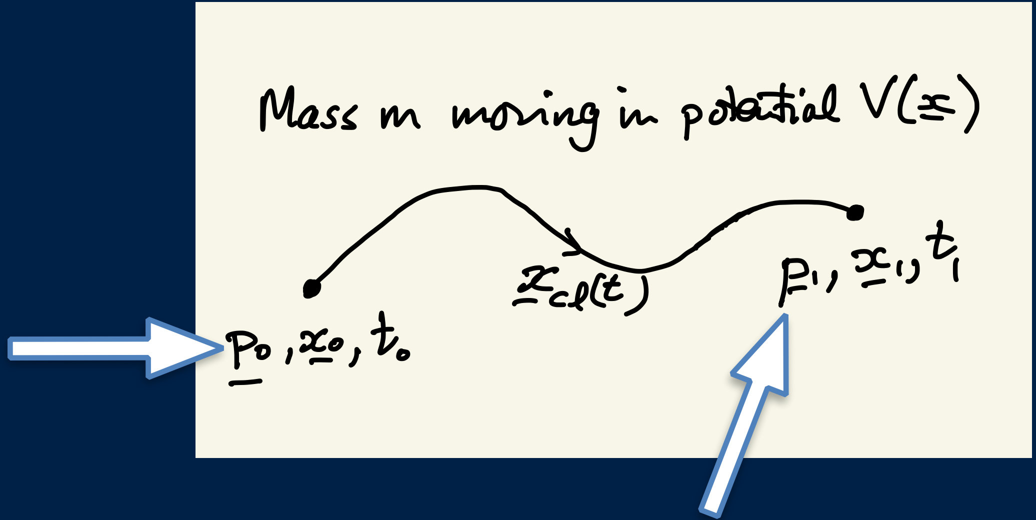
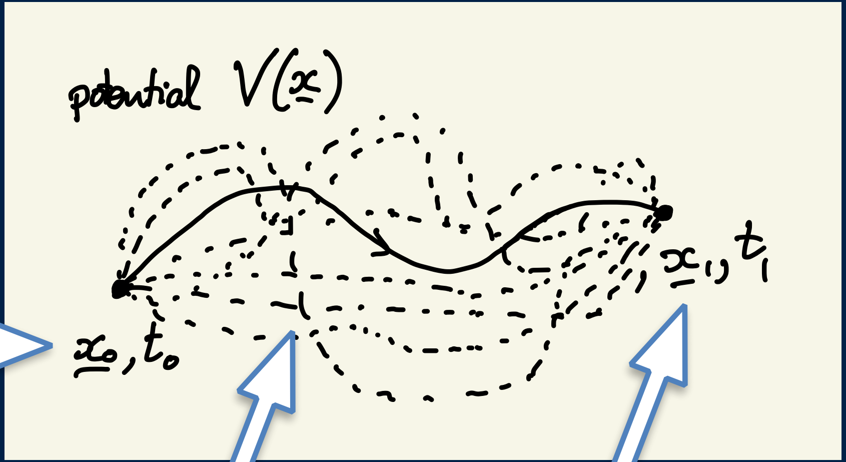
Particle Mechanics



Classically $\hbar = 0$, so we can know $\mathbf{x}_{cl}(t)$ and $\mathbf{p}_{cl}(t)$ but quantum mechanically this is not allowed so …



$= \langle \mathbf{x}_1, t_1 | \mathbf{x}_0, t_0 \rangle$

Momentum not specified

All paths possible

Momentum not measured

Feynman Path Integral: sum over all paths gives amplitude

$$\langle \mathbf{x}_1, t_1 | \mathbf{x}_0, t_0 \rangle = \int D\mathbf{x} \exp\left(\frac{i}{\hbar} \int_{t_0}^{t_1} dt \; \tfrac{1}{2} m \dot{\mathbf{x}}^2 - V(\mathbf{x}) \right)$$

Classical physics $\hbar \to 0$ integral dominated by stationary point

$$\delta S = \left(\int_{t_0}^{t_1} dt \ \tfrac{1}{2} m (\dot{\mathbf{x}}_{cl} + \delta \dot{\mathbf{x}})^2 - V(\mathbf{x}_{cl} + \delta \mathbf{x}) \right) - \left(\int_{t_0}^{t_1} dt \ \tfrac{1}{2} m \dot{\mathbf{x}}^2 - V(\mathbf{x}) \right)$$

$$= \int_{t_0}^{t_1} dt \ m \dot{\mathbf{x}}_{cl} \, . \frac{d}{dt} \delta \mathbf{x} - \nabla V(\mathbf{x}_{cl}) \, . \, \delta \mathbf{x}$$

$$= \left[m \dot{\mathbf{x}}_{cl} \, . \, \delta \mathbf{x} \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \ \left(-m \ddot{\mathbf{x}}_{cl} - \nabla V(\mathbf{x}_{cl}) \right) . \, \delta \mathbf{x}$$

↓ 0

↓ $m \ddot{\mathbf{x}}_{cl} = - \nabla V(\mathbf{x}_{cl})$

Equation of motion

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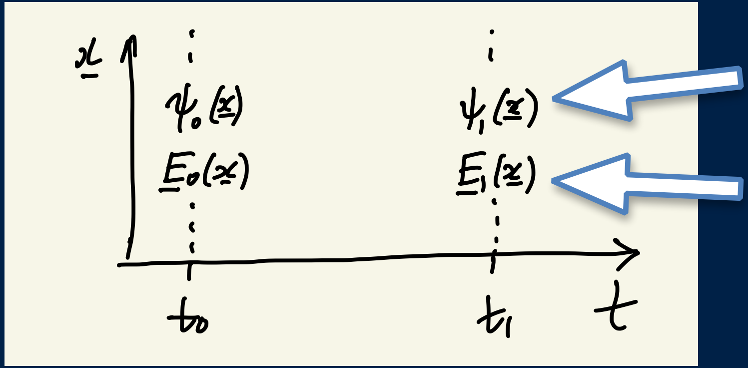
QED



matter field

electric field

$$= \langle \mathbf{E}_1(\mathbf{x}), \psi_1(\mathbf{x}), t_1 \,|\, \mathbf{E}_0(\mathbf{x}), \psi_0(\mathbf{x}), t_0 \rangle$$

$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla V + \partial_t \mathbf{A}$ can be written as the field strength tensor

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where $A_\mu = (V, \mathbf{A})$ and $\mu, \nu \in 0,1,2,3$

$F_{\mu\nu}$ and A_μ transform covariantly under Lorentz transformations

Gauge invariance: when $A_\mu \to A_\mu + \partial_\mu \chi$ we see that $F_{\mu\nu} \to F_{\mu\nu}$

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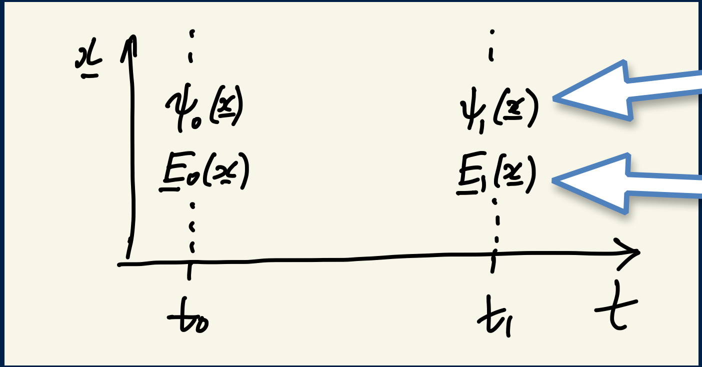
QED



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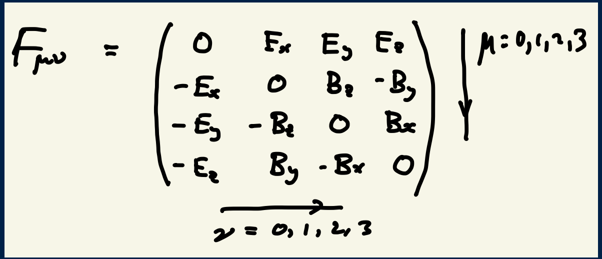

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$F_{\mu\nu}$ and A_μ transform covariantly under Lorentz transformations

Gauge invariance: when $A_\mu \to A_\mu + \partial_\mu \chi$ we see that $F_{\mu\nu} \to F_{\mu\nu}$

Feynman PI: sum over all interpolating field configurations

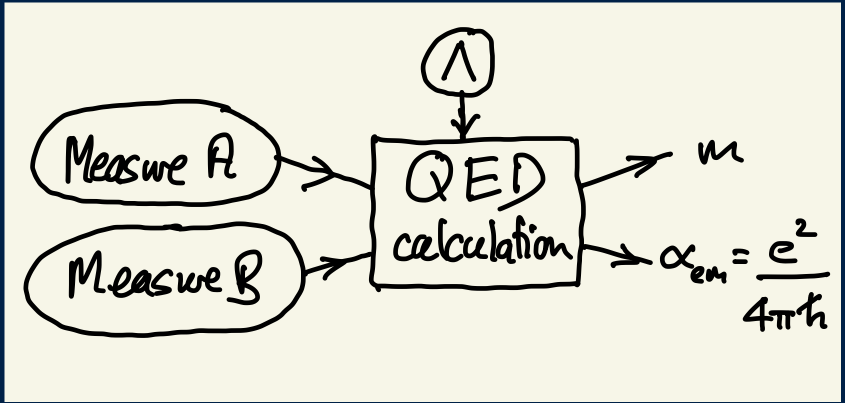
$$\langle \mathbf{E}_1(\mathbf{x}), \psi_1(\mathbf{x}), t_1 \,|\, \mathbf{E}_0(\mathbf{x}), \psi_0(\mathbf{x}), t_0 \rangle =$$

$$\int DAD\psi \exp\left(\frac{i}{\hbar} \int_{t_0}^{t_1} dt \int d^3\mathbf{x} \ \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(\gamma^\mu(\partial_\mu - ieA_\mu) - m)\psi \right) \right)$$

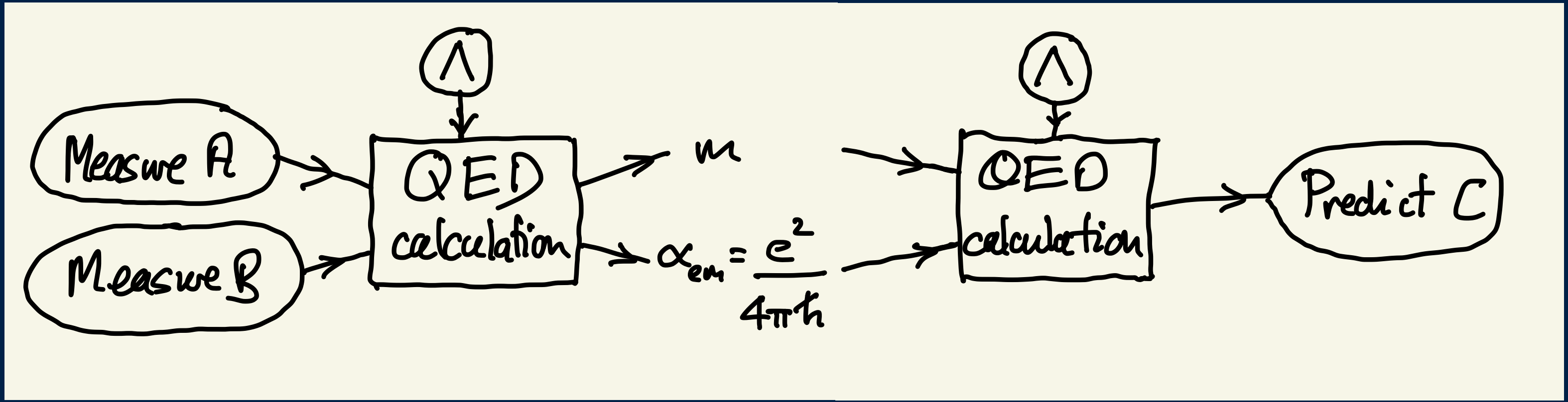
$\hbar \to 0$ stationary point gives Maxwell's equations $\partial^\mu F_{\mu\nu} = j_\nu$

$\hbar \neq 0$ we get Quantum Electrodynamics (QED) —

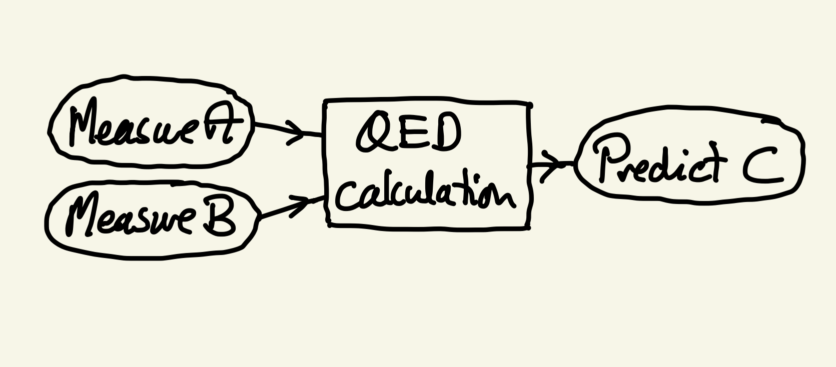
in principle the FPI finds any amplitude in terms of $e, m, \hbar$ and the boundary conditions, in practice hard work!



Λ
Measure A
Measure B
QED calculation
m
$\alpha_{em} = \frac{e^2}{4\pi\hbar}$


Λ
Λ
Measure A
Measure B
QED calculation
m
$\alpha_{em} = \frac{e^2}{4\pi\hbar}$
QED calculation
Predict C

But actually….



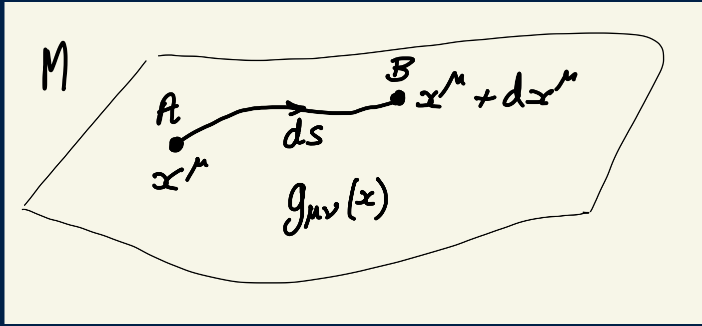
QED is *renormalizable* —

It predicts unambiguous relationships between measurables and expansion in α_{em} works spectacularly well eg

$$a_e = \frac{g-2}{2} = \begin{array}{ll} 0.001\ 159\ 652\ 180\ 73\ (28) & \text{experiment} \\ 0.001\ 159\ 652\ 181\ 60\ (23) & \text{theory} \end{array}$$

General Relativity

Space M of fixed topology, and metric $g_{\mu\nu}$



Proper distance $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Euclidean plane $ds^2 = dx^2 + dy^2$

We can change coordinate systems eg Euclidean to plane polar; the *coordinates* of A and B change but the proper distance does not

Re-parametrization: $x^\mu \rightarrow x'^\mu(x), \quad g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x')$

but $ds^2 \rightarrow ds^2$

Proper distance is not the only frame independent characteristic of $M, g_{\mu\nu}$

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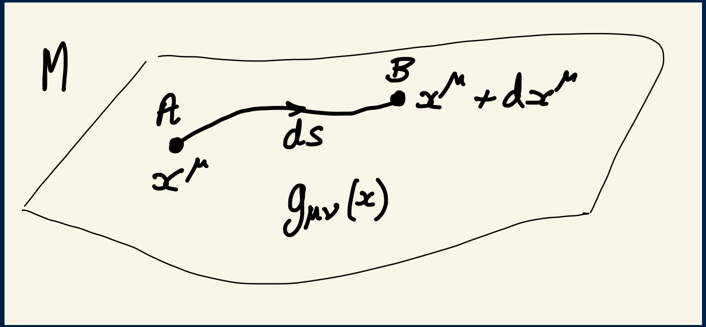

Proper distance $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Two-sphere $ds^2 = k^2(d\theta^2 + \sin^2\theta d\phi^2)$

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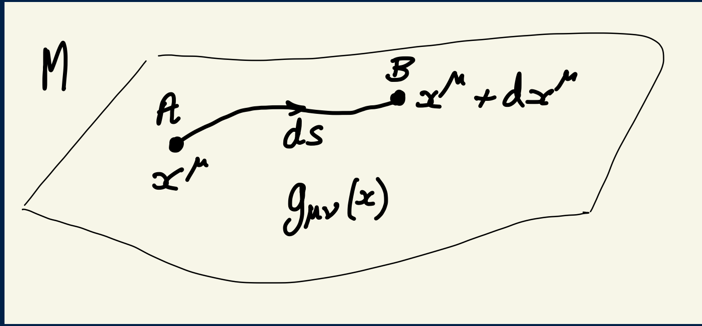

Proper distance $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$

Minkowski space-time $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = dt^2 - d\mathbf{x}^2$

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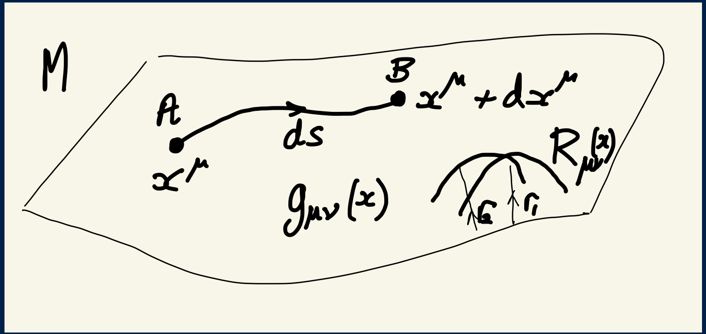
Proper distance is not the only frame independent characteristic of $M, g_{\mu\nu}$

Intrinsic curvature

$$R^{\rho}_{\mu\nu\lambda}, \quad R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda}, \quad R = R_{\mu\nu} g^{\mu\nu}$$

Euclidean plane $\quad R = 0$



$g_{\mu\nu}$ is the dynamical degree of freedom + curvature of space-time is generated by mass/energy leading to Einstein's equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda_c g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Non-linear partial differential equations for the metric; specify initial conditions and solve! Excellent agreement with observation and experiment

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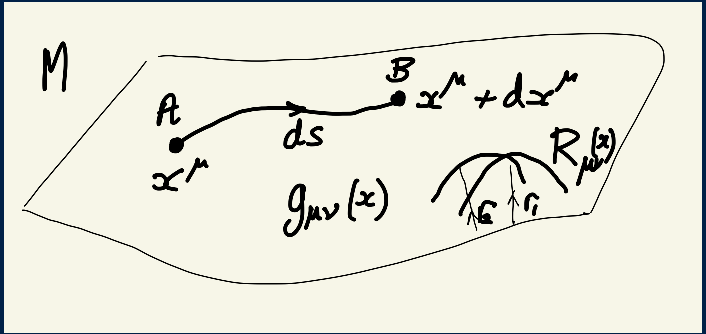
Two-sphere $\quad R = \frac{1}{r_1 r_2} = k^{-2}$



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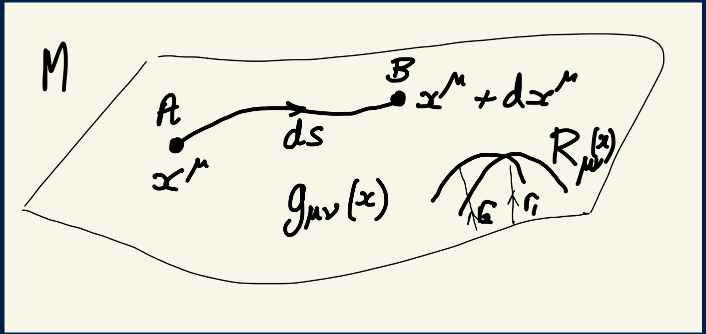

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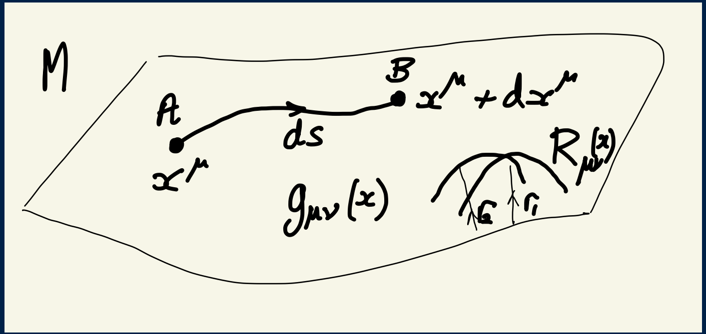

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stress-energy tensor

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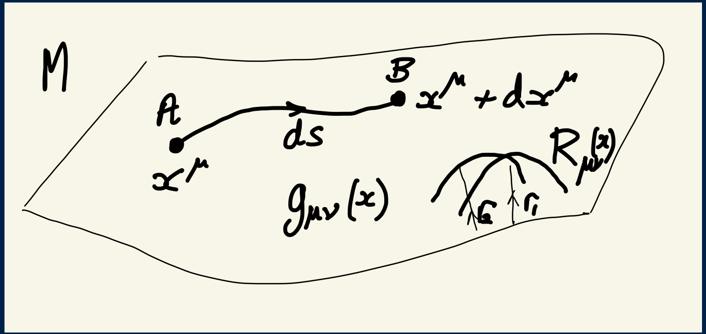

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stress-energy tensor

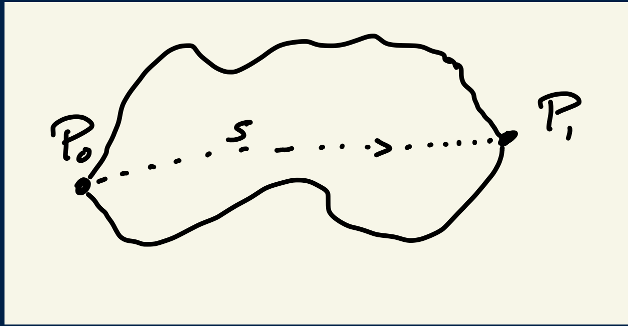
cosmological term

Non-linear partial differential equations for the metric; specify initial conditions and solve! Excellent agreement with observation and experiment

Quantum Gravity?

Physical amplitudes must be reparametrization invariant …

$$\langle P_1, s \,|\, P_0, 0\rangle =$$



universe exists for proper time s and

FPI is a sum over all such metrics ?

$$\int DgD\psi \exp\left(\frac{i}{\hbar}\int d^4x\sqrt{-g}\;\left(\frac{1}{16\pi G}(R-2\Lambda_c) + \text{matter} + \text{boundary terms}\right)\right)$$

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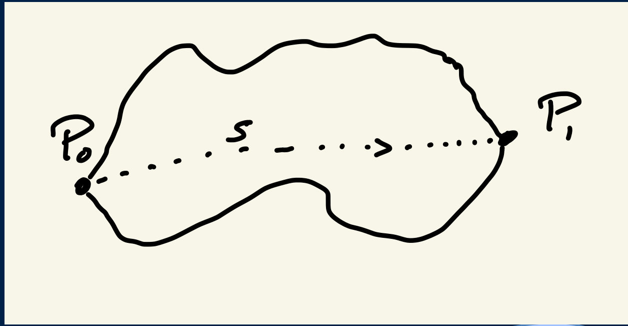

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Einstein-Hilbert action $\hbar \to 0$ stationary point gives Einstein's equations

Quantum Gravity?

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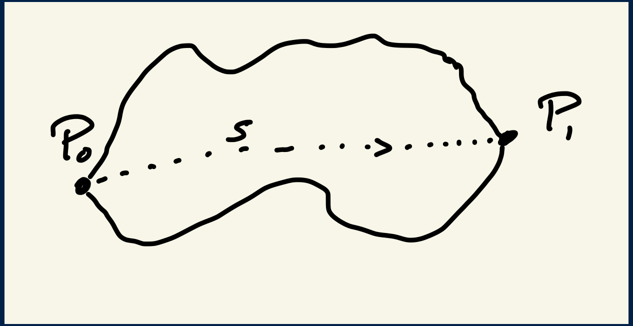

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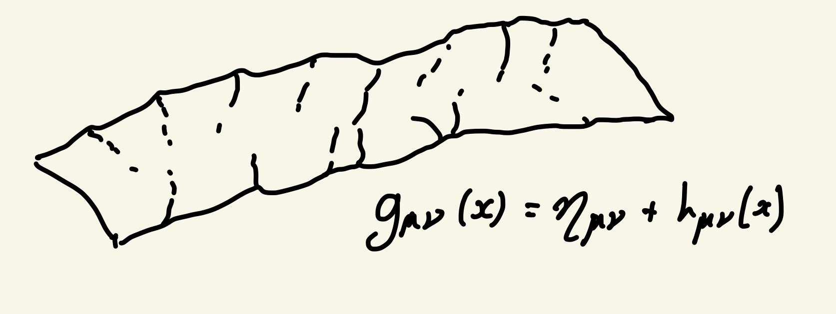
Quantum Field Theory in a fixed background space-time

What does $\int Dg$ mean?

First attempt — copy QED with a perturbation expansion in G

gravity is 'weak' so perhaps…

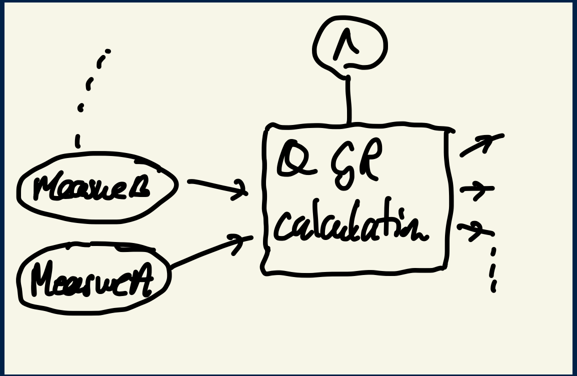


$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$ ⇨ $\partial^\mu \partial_\mu \bar{h}_{\lambda\rho} = -16\pi G T_{\mu\nu}$ ⇨ gravitational waves

In Q-GR fine structure constant $\alpha_{GR} = \frac{G\Lambda^2}{\hbar}$ is the expansion parameter

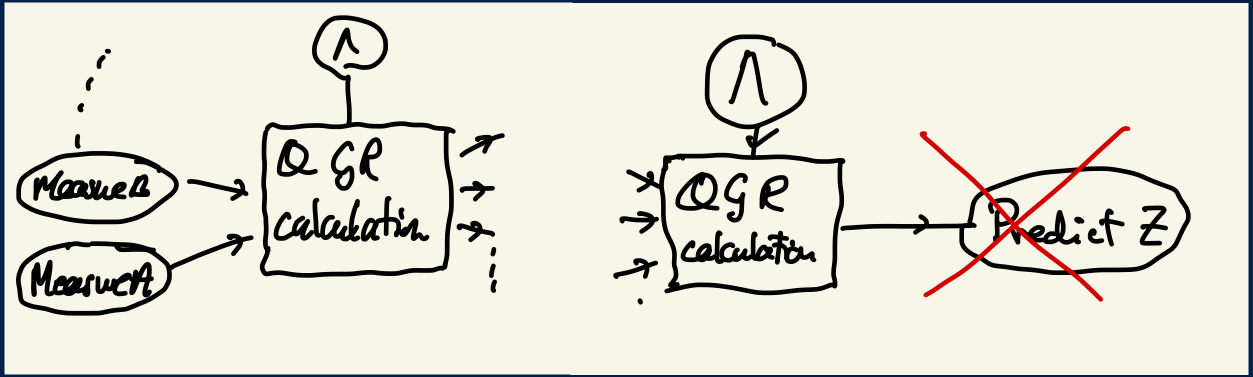
$\Lambda = m_e, \quad \alpha_{GR} \approx 10^{-46}$ $\qquad$ $\Lambda \approx 10^{23} m_e \approx 10^{-7}\mathrm{kg}, \quad \alpha_{GR} \approx 1$

weak!! $\qquad$ hmmm….



Q-GR is *not* renormalizable —

- Keep Λ, work in a regime where it is small — 'effective field theory', learn a lot but it's not the final solution
- String theory — contains $h_{\mu\nu}$ and a consistent minimum distance scale, but lots of other degrees of freedom



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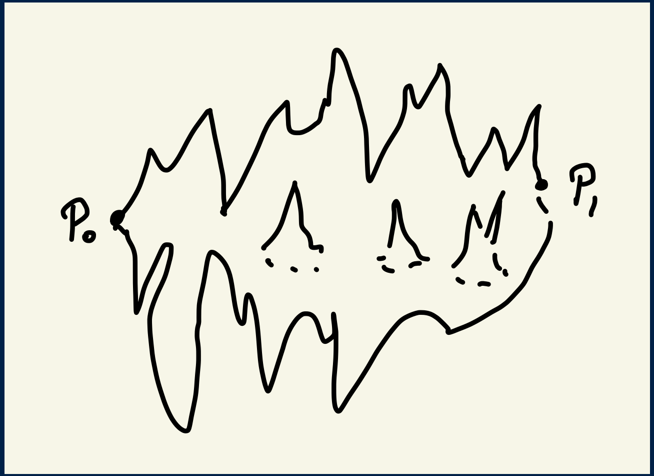
What does $\int Dg$ mean?

Second attempt —

metric democracy…



4-dimensional manifolds are only partially charted territory —

it's a very hard problem, so let's look at a toy model …

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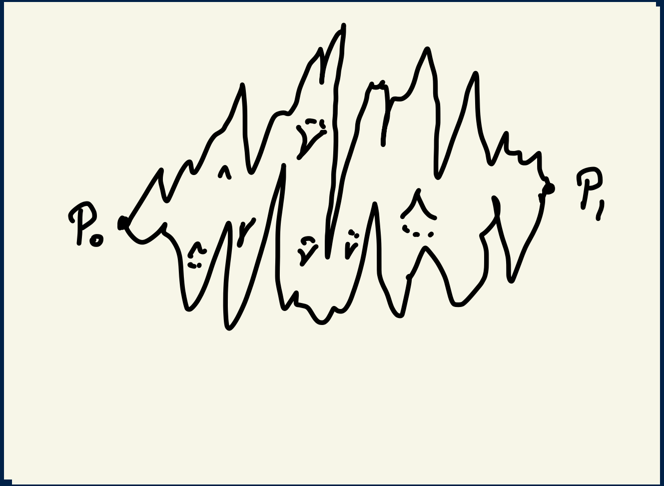

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$\Longrightarrow$ topology democracy?

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What does $\int Dg$ mean?

Second attempt —

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$\Longrightarrow$ topology democracy?

Wormholes!

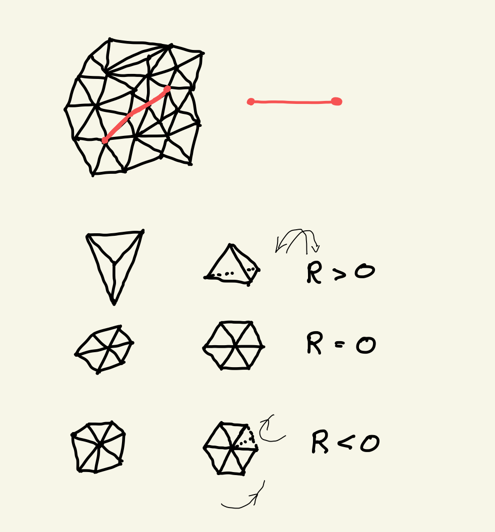
Splits!

Joins!

4-dimensional manifolds are only partially charted territory —

it's a very hard problem, so let's look at a toy model …

Graph of equilateral triangles…



graph distance ~ geodesic distance

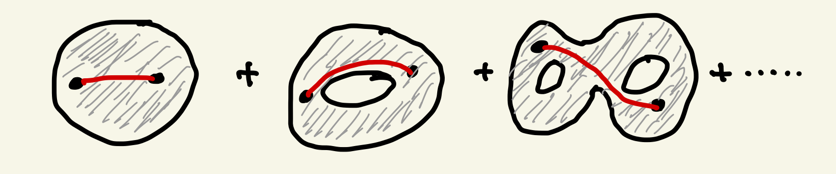
3-fold vertex ~ place of positive curvature

6-fold vertex ~ place of zero curvature

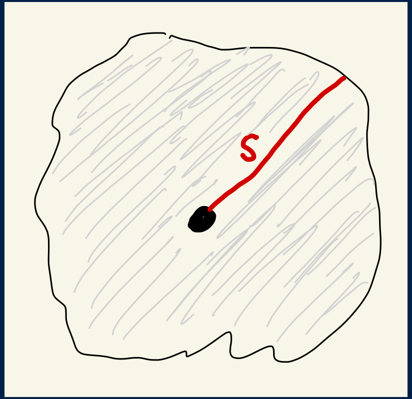
7-fold vertex ~ place of negative curvature

Idea… Fix topology, $\sum_{T} \sim \int Dg$, then sum over genus

$$\langle P_1, s \,|\, P_0, 0\rangle = \sum_{T}$$



What do our ‘universes’ look like?



$$\text{Volume}(s < s_0) \propto s_0^4$$

A long, long way from flat 2d space!

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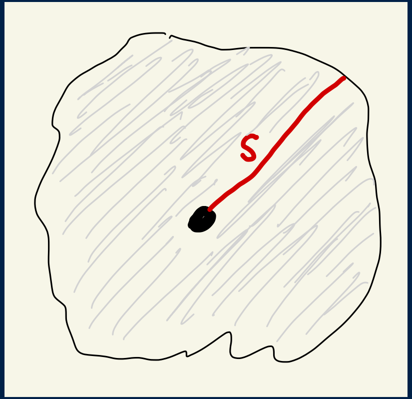

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A long, long way from flat 2d space!

It is extraordinarily non-trivial that our quantum universe is so very four-dimensional locally and globally