Breaking through the quantum barrier

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6 February 2016

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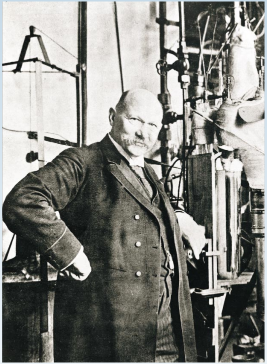
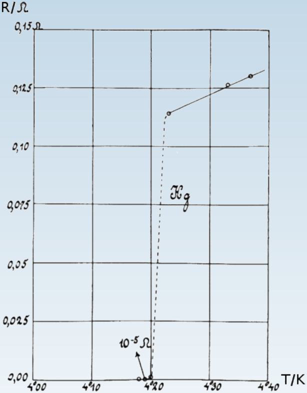
Why should we want to break through the "quantum barrier" and what is it in the first place?

Superconductivity

- 1911 Discovery of superconductivity 1913



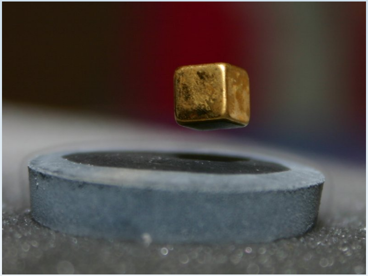
Heike Kamerlingh Onnes



- 1920s and 30s: quantum mechanics, Schrödinger equation 1933
- 1957 explanation given by Bardeen, Cooper and Schrieffer 1972

High-temperature Superconductivity

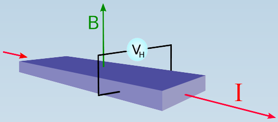
- 1986 High temperature superconductivity in $La_{1.85}Ba_{0.15}CuO_4$ (Bednorz and Müller  1987)



World record:
138 K (-135 °C)

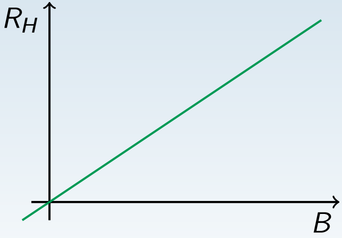
- 20?? explanation 

Quantum Hall Effect

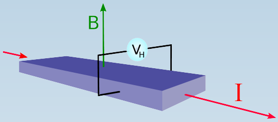


$$R_H \propto B$$

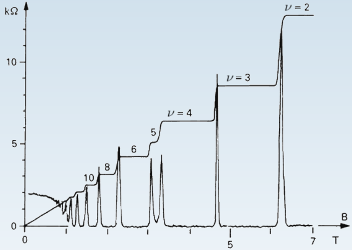
classical case:



Quantum Hall Effect



$$R_H = \frac{h}{\nu e^2}$$

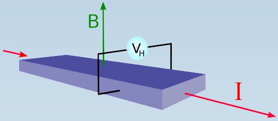


K. v. Klitzing, G. Dorda, M. Pepper, Phys. Rev. Lett. 45, 494 (1980)

1980 Integer Quantum Hall Effect

1985

Quantum Hall Effect



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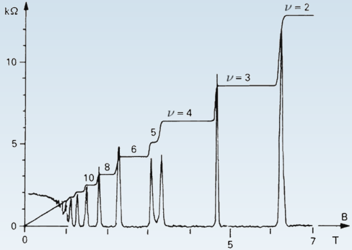

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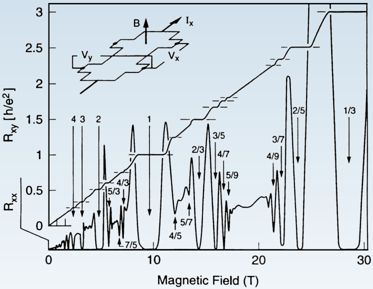

1982 Fractional Quantum Hall Effect

 1998

Why such effects are incredibly hard to describe

The Schrödinger Equation

$$\left(-\sum_{i=1}^{M}\frac{\hbar^2}{2m_i}\nabla_i^2+\sum_{i>j=1}^{M}V_{\mathrm{int}}(\mathbf{r}_i-\mathbf{r}_j)+\sum_{i=1}^{M}V_{\mathrm{ext}}(\mathbf{r}_i)\right)\psi_n(\mathbf{r}_1,\ldots,\mathbf{r}_M)$$
$$=E_n\,\psi_n(\mathbf{r}_1,\ldots,\mathbf{r}_M)$$

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$$\underbrace{\left(-\sum_{i=1}^{M}\frac{\hbar^2}{2m_i}\nabla_i^2+\sum_{i>j=1}^{M}V_{\text{int}}(\mathbf{r}_i-\mathbf{r}_j)+\sum_{i=1}^{M}V_{\text{ext}}(\mathbf{r}_i)\right)}_{\text{Hamiltonian }\hat{H}}\psi_n(\mathbf{r}_1,\ldots,\mathbf{r}_M)$$

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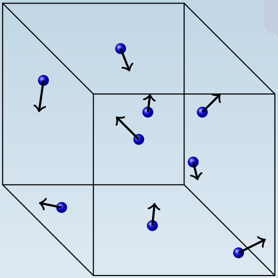
The quantum barrier



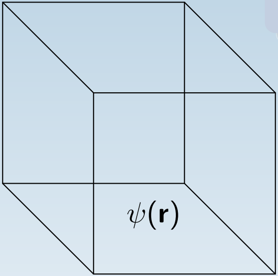
M identical particles in a box

Classical system

$$\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M\}$$
$$\{\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots, \dot{\mathbf{r}}_M\}$$

$\rightarrow \; 6M$ numbers $\in \mathbb{R}$

The quantum barrier



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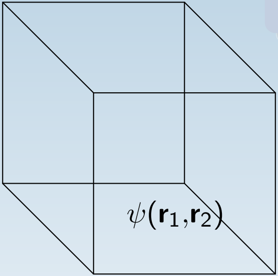
Quantum mechanical problem, $M = 1$

$$\psi(\mathbf{r}) \in \mathbb{C},\ p(\mathbf{r}) = |\psi(\mathbf{r})|^2$$

The quantum barrier



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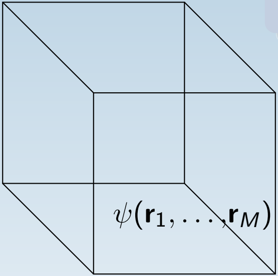
$$\rightarrow \; 6M \text{ numbers} \in \mathbb{R}$$

Quantum mechanical problem, $M = 2$

$$\psi(\mathbf{r}_1, \mathbf{r}_2) \in \mathbb{C},\ p(\mathbf{r}_1, \mathbf{r}_2) = |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2$$

The quantum barrier



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Quantum mechanical problem, *M* arbitrary

$$\psi(\mathbf{r}_1, \ldots, \mathbf{r}_M) \in \mathbb{C}, \; p(\mathbf{r}_1, \ldots, \mathbf{r}_M) = |\psi(\mathbf{r}_1, \ldots, \mathbf{r}_M)|^2$$

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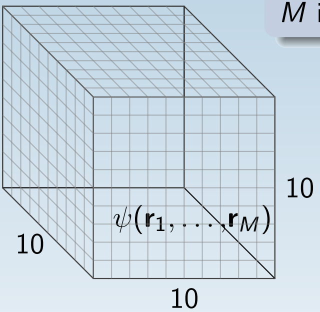

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$\rightarrow$ $6M$ numbers $\in \mathbb{R}$

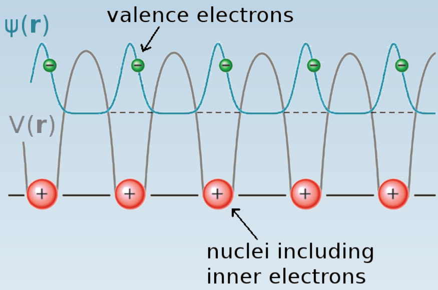
$$10 \times 10 \times 10 = 1000$$

Quantum mechanical problem

$$\psi(\mathbf{r}_1, \ldots, \mathbf{r}_M) \in \mathbb{C}, \; p(\mathbf{r}_1, \ldots, \mathbf{r}_M) = |\psi(\mathbf{r}_1, \ldots, \mathbf{r}_M)|^2$$

1000^M numbers $\in \mathbb{C}$ (impossible for $M > 4$)

Physical motivation for a lattice model



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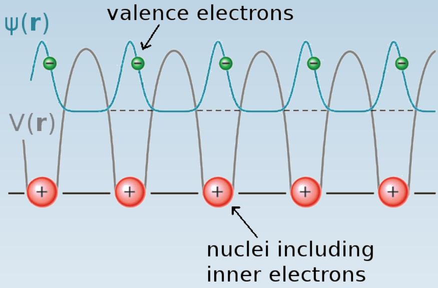

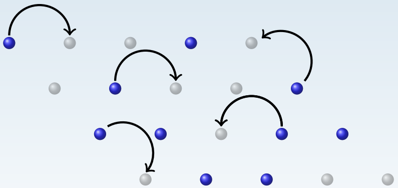

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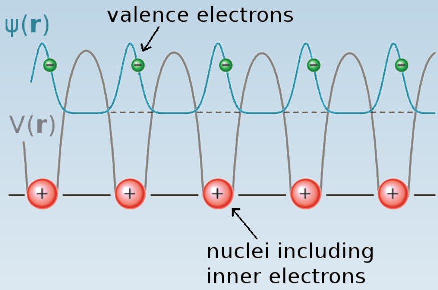

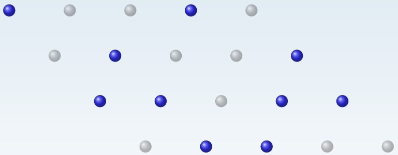

Representation of a state

$\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M) \rightarrow |\psi\rangle$

Orthonormal basis:

$\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_k\rangle, \ldots\}$

$|\psi\rangle = \sum_k c_k |\varphi_k\rangle$

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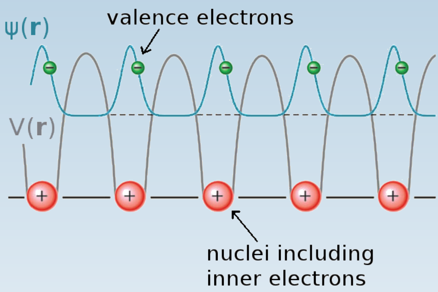

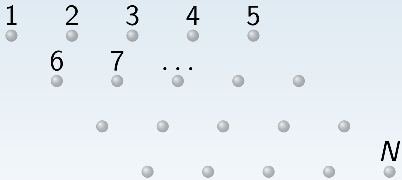

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$|000\ldots0\rangle$

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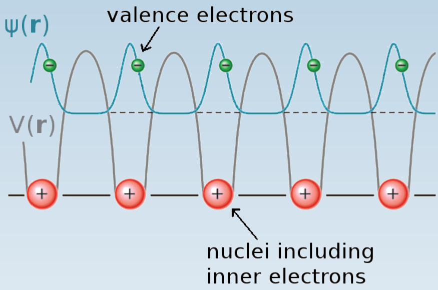

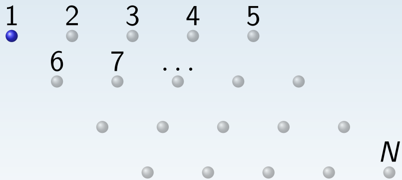

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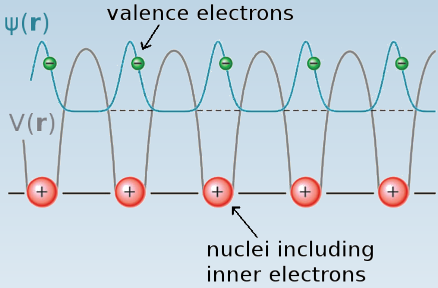

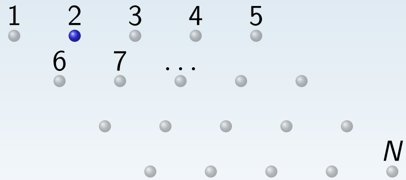

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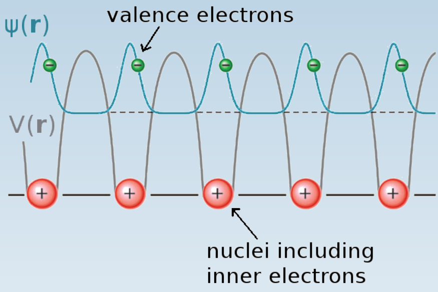

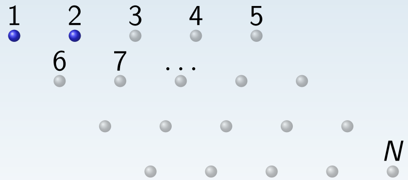

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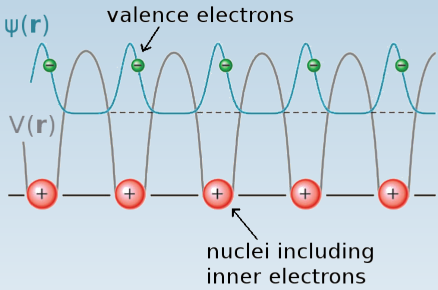

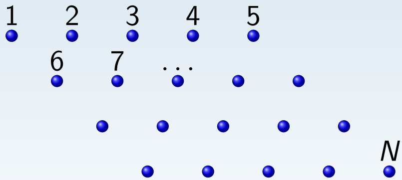

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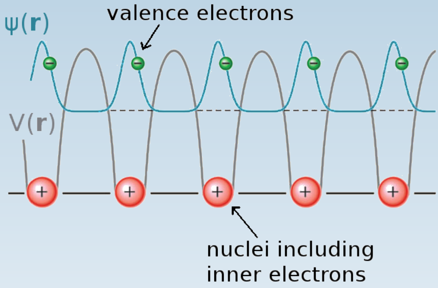

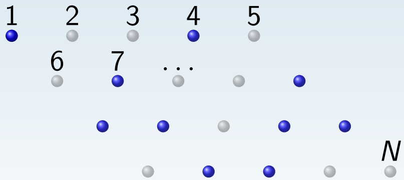

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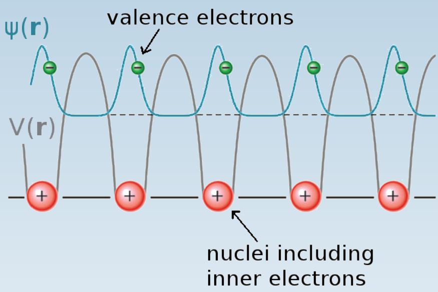
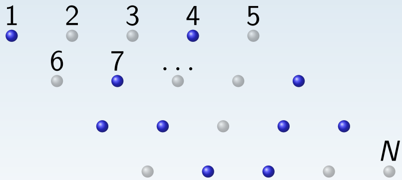
Physical motivation for a lattice model





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$\vert 100\ldots0\rangle$	$\vert\varphi_2\rangle$
$\vert 010\ldots0\rangle$	$\vert\varphi_3\rangle$
$\vert 110\ldots0\rangle$	$\vert\varphi_4\rangle$
$\ldots$	
$\vert 111\ldots1\rangle$	$\vert\varphi_{2^N}\rangle$

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Problem

Calculations exponentially hard
- both in the continuum and on the lattice!

Reduction to polynomial scaling:

1. naïve ansatz: Hartree-Fock
2. Quantum Monte Carlo

Based on physical insight

3. sophisticated ansatz:
Tensor Network States

Idea 1: Naïve approximation ansatz - Hartree-Fock

Idea: no correlations

wave function is built of independent states for the individual electrons (= orbitals)

- exact solution for **non-interacting** electrons
- otherwise approximation of the true ground state

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Advantage: easy to use

explains

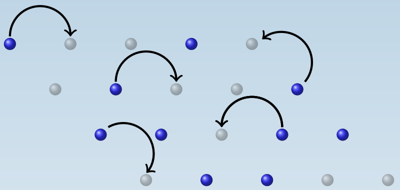
- low temperature super-conductivity
- integer Quantum Hall Effect

Disdvantage: bad approximation

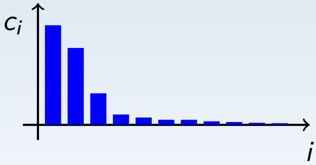
fails at

- high temperature super-conductivity
- fractional Quantum Hall Effect

Idea 2: Quantum Monte Carlo



importance sampling



$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N = 0}^{1} c_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle$$

- does not work for fermionic systems (sign problem)

Idea 3 based on profound physical insight

Ground states of realistic Hamiltonians fulfil the **Area Law of Entanglement**.

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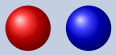
Entanglement

Consider the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$



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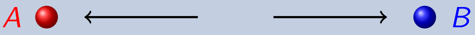

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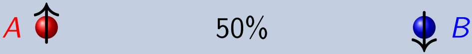

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A  50%  *B*

Entanglement!

No such effect for $|\psi'\rangle = |\uparrow\uparrow\rangle$.

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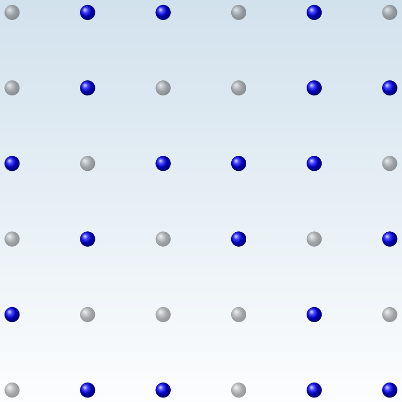
$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N = 0}^{1} c_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle$$



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Area Law (realistic Hamiltonians)

$\mathrm{Entanglement}(A, B) \propto \partial A$

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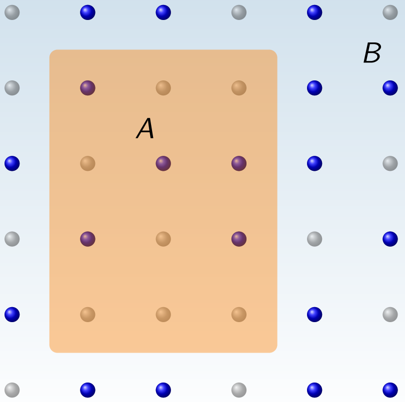
generic state:

$\mathrm{Entanglement}(A, B) \propto \mathrm{num}(A)$

exp. scaling $\xrightarrow{\text{Area Law}}$ poly. scaling



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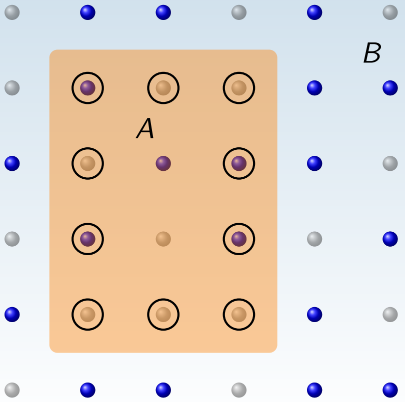


Idea 3 based on profound physical insight

Area Law (realistic Hamiltonians)

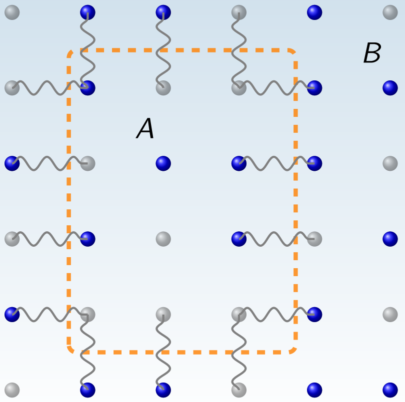
$\mathrm{Entanglement}(A, B) \propto \partial A$

$$|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N = 0}^{1} c_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle$$

generic state:

$\mathrm{Entanglement}(A, B) \propto \mathrm{num}(A)$

exp. scaling $\xrightarrow{\text{Area Law}}$ poly. scaling



Idea 3

Construct states that fulfil the Area Law:

Tensor Network States

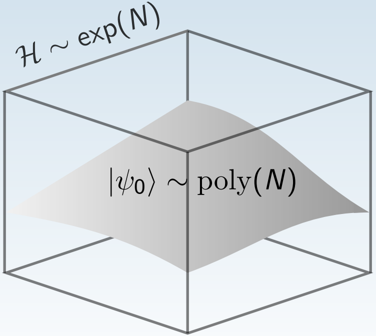
Overcoming the exponential scaling problem

Problems of previous approximation schemes

1. Hartree-Fock: Limited applicability
2. Quantum Monte Carlo: sign problem

Area Law

- strong restriction on physical ground states
- Tensor Network States



Construction of Tensor Network States

Theorem

Tensor Network States fulfil the Area Law:

- every site gets a tensor $A^{i}_{\alpha_1\alpha_2\ldots\alpha_z}$
- **bond dimension**: $\alpha = 1, \ldots, D, \; i = 0, 1$

Construction of Tensor Network States

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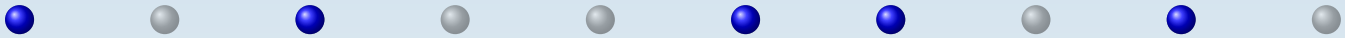
In 1D:



$A^{i_1}_{\alpha\beta} \quad B^{i_2}_{\beta\gamma} \quad C^{i_3}_{\gamma\delta} \quad \cdots \quad Z^{i_N}_{\omega\alpha}$

Construction of Tensor Network States

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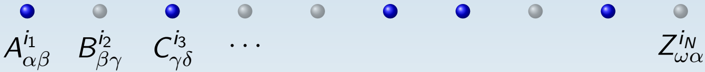


Matrix Product States

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=0}^{1} \underbrace{A^{i_1}_{\alpha\beta} B^{i_2}_{\beta\gamma} C^{i_3}_{\gamma\delta} \ldots Z^{i_N}_{\omega\alpha}}_{c_{i_1 i_2 i_3 \ldots i_N}} |i_1 i_2 i_3 \ldots i_N\rangle$$

- number of parameters $= 2D^2 N$

Performance of Matrix Product States

Ground state energies

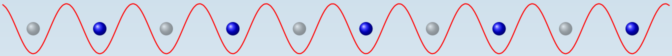
- extremely high accuracies
  e.g., spin-1/2 anti-ferromagnetic Heisenberg chain:

	exact	numerical
E_0	-0.4431471806	-0.4431471825

Time Evolution

- optical lattice
- time-dependent Schrödinger equation

$$|\psi_{\text{init}}\rangle = |0101010101\rangle$$



Time Evolution

- optical lattice
- time-dependent Schrödinger equation

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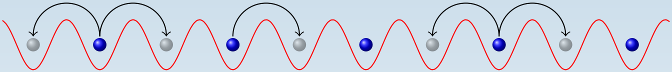
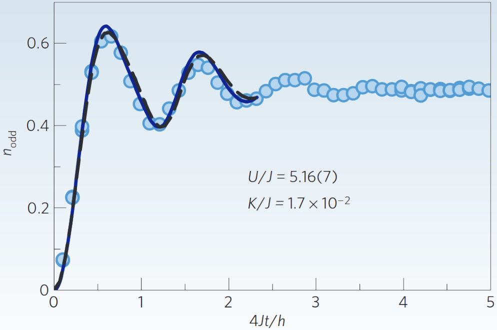
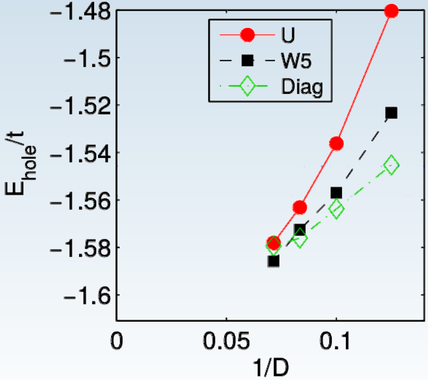




S. Trotzky, Y-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Nature Physics **8**, 325 (2012).

Two dimensional Tensor Network States

- given by tensors $A^i_{\alpha\beta\gamma\delta}$, $\alpha = 1, \ldots, D$
- efficient algorithms are being developed
- promising results for the Hubbard model (high temperature superconductivity):



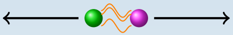
P. Corboz, T. M. Rice, and M. Troyer, PRL 2014

2D Tensor Network States in High Energy Physics

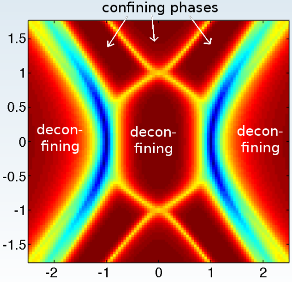
1. Simulation with
   - classical quarks
   - quantum fields

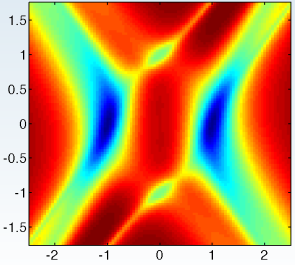
2. Simulation with
   - quantum quarks (no sign problem!)
   - quantum fields

Confinement:



$$E = \infty$$





E. Zohar, M. Burrello, T. B. Wahl, and J. I. Cirac, Ann. Phys. **363**, 385 (2015)

Summary

Description of Quantum Matter

solving the Schrödinger equation: exponentially hard

Tensor Network States: use Area Law $\Rightarrow$ polynomial overhead

- no sign problem
- huge successes in 1D
- first impressive results in 2D