Gravitational Waves

James Binney

Rudolf Peierls Centre for Theoretical Physics

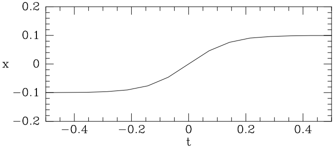
6 May 2017

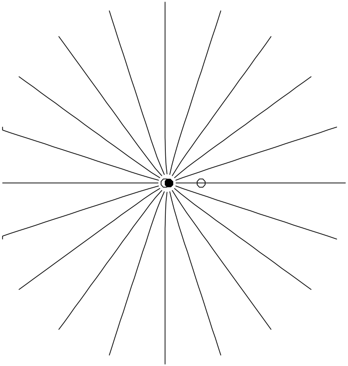
Outline

Outline

Underlying physics

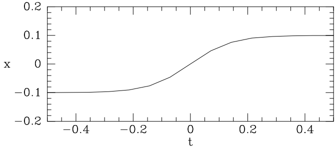
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

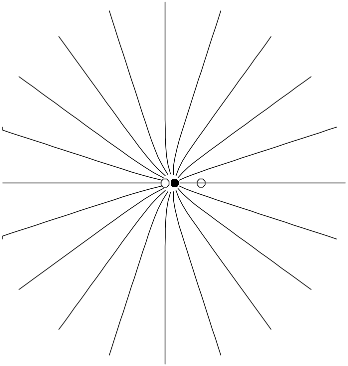




Underlying physics

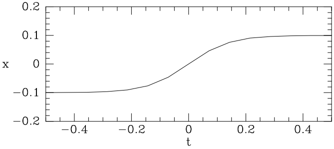
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

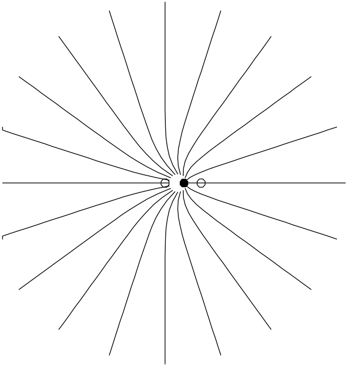




Underlying physics

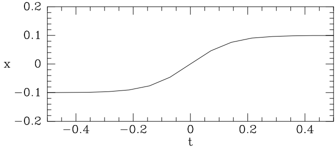
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

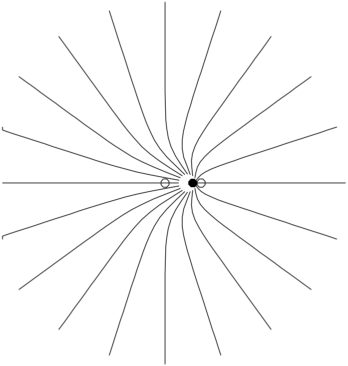




Underlying physics

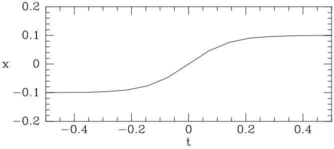
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

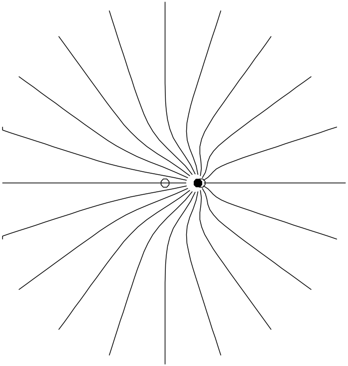




Underlying physics

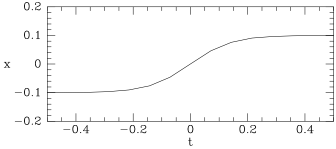
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

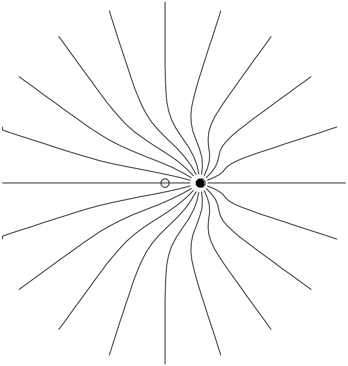




Underlying physics

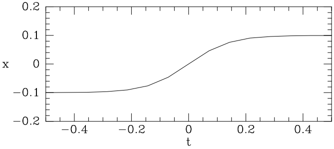
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

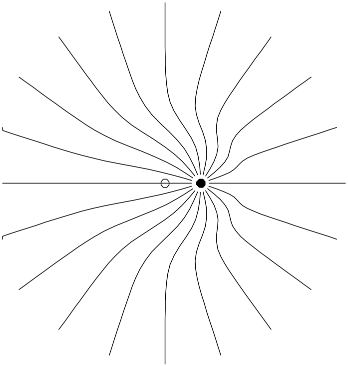




Underlying physics

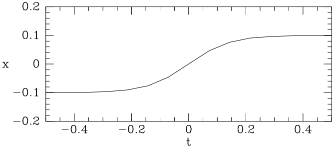
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

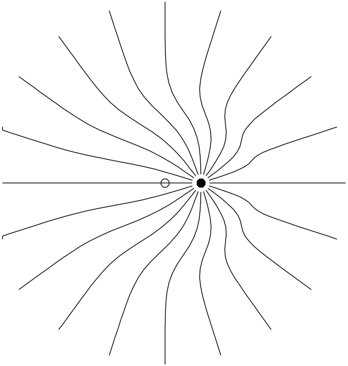




Underlying physics

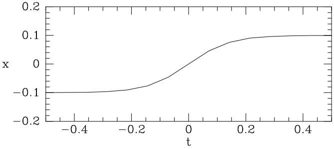
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

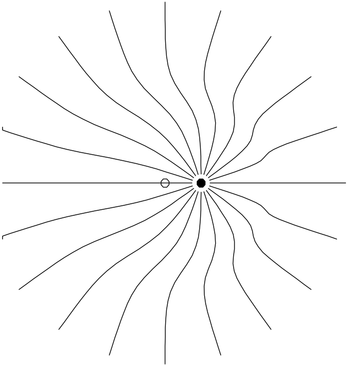




Underlying physics

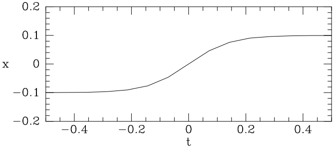
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

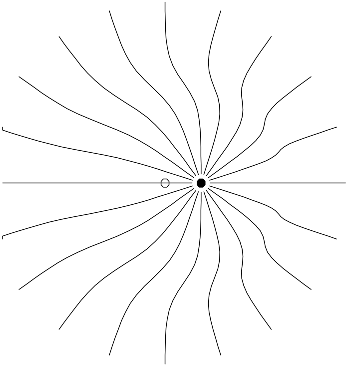




Underlying physics

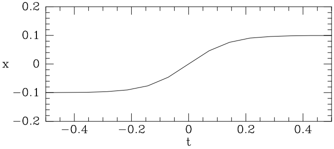
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

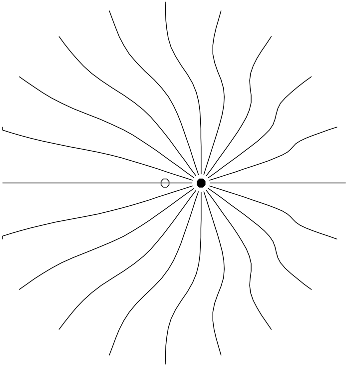




Underlying physics

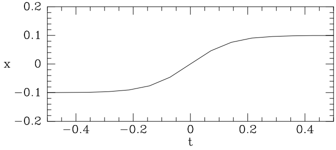
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

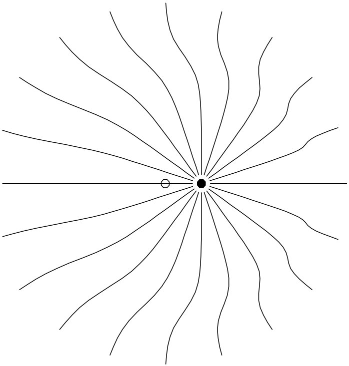




Underlying physics

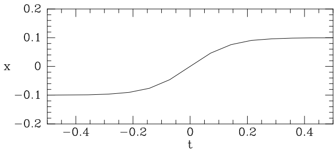
- Relativistic covariance is a fundamental principle:
    - no communication faster than c
- It guarantees the existence of emag & gravitational waves

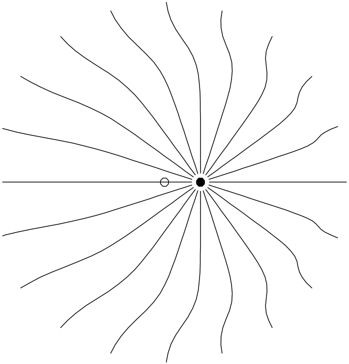




Underlying physics

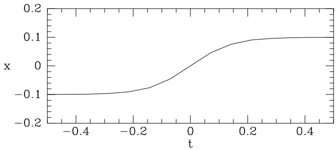
- Relativistic covariance is a fundamental principle:
  - no communication faster than c
- It guarantees the existence of emag & gravitational waves

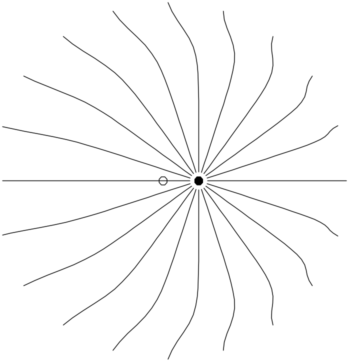




Underlying physics

- Relativistic covariance is a fundamental principle:
    - no communication faster than c
- It guarantees the existence of emag & gravitational waves





Outline

Preliminaries

- We have hands-on experience of only a tiny corner of gravity
  - So we cannot understand gravitational waves in a similar physical way to emag waves – try explaining emag waves to someone who knows only electrostatics!
- We have to rely on maths
  - exploit strong parallels with emag
- Notation:

$$x^\mu \equiv (ct, x, y, z) \qquad x_\mu x^\mu \equiv \sum_{\mu=0}^{3} x_\mu x^\mu \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

Maths of wave generation

- The emag field is quantified by the Maxwell field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ & 0 & B_z & -B_y \\ \text{-ditto} & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

- Half of Maxwell's eqns ($\epsilon_0\mu_0 = c^{-2}$):

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad \begin{cases} \nu = 0 & \nabla\cdot\mathbf{E} = \rho/\epsilon_0 \\ \nu \neq 0 & \nabla\times\mathbf{B} - c^{-2}\dot{\mathbf{E}} = \mu_0\mathbf{j} \end{cases}$$

- In terms of the emag 4-potential $A_\mu = (\phi/c, A_x, A_y, A_z)$: $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ so

$$\mu_0 j_\nu = \partial^\mu F_{\mu\nu} = \partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu$$

- In radiation gauge $\partial^\mu A_\mu = 0$ so

$$\mu_0 j_\nu = \Box A_\nu \quad \text{where} \quad \Box \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Radiation

- $\Box\phi = 0$ has a spherical solution

$$\phi(r,t) = \text{const} \times \frac{\sin[\omega(t - r/c)]}{r}$$

- so

$$A^\mu = a^\mu \frac{\sin[\omega(t - r/c)]}{r}$$

$$\Rightarrow \quad E^\mu \sim \frac{\partial A^\mu}{\partial t} = \omega a^\mu \frac{\cos[\omega(t - r/c)]}{r} \quad B \sim \text{ditto}/c$$

so E flux (Poynting vector) $\mathbf{N} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} \sim \frac{1}{r^2}$

- The disturbance detaches from its source and carries energy to infinity

Outline

Now GR

- The emag $\to$ GR correspondence:

$$\begin{array}{lll} A_\mu & \to & g_{\mu\nu} \qquad \text{'metric'}: \ \mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ F_{\alpha\beta} & \to & \Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) \end{array}$$

- The Christoffel symbol Γ is proportional to the gradient of **g**
- It encodes the gravitational field:
  - the eqn of motion of the 4-velocity **u** for a particle of rest mass m_0 & charge q is

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = -\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta + \frac{q}{m_0}F^\mu{}_\alpha u^\alpha$$

principle of equivalence: $1 = m_{\mathrm{gravitaional}}/m_{\mathrm{inertial}}$

- Emag wave ripples in **A**; gravitational wave ripples in **g**

Gauge conditions

- To get $\Box\mathbf{A} = \mu_0\mathbf{j}$ we needed to adopt the radiation gauge
- A gauge condition doesn't change *physics* but it can greatly simplify the *maths*
- In GR, gauge condition $\leftrightarrow$ choice of coordinates
- Far from the source the ripples will be small ($\sim 10^{-21}$!) so we can assume space-time is almost flat. Then there are coordinate systems in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \begin{cases} |\mathbf{h}| \ll 1 \\ \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \end{cases}$$

Harmonic coordinates

- We narrow our choice of coordinates to harmonic coordinates by requiring

$$g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta} = 0$$

$$\Rightarrow g^{\alpha\beta}g^{\mu\nu}\left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right) = 0$$

cf. the radiation gauge condition $\partial_\mu A^\mu = 0$

- In this gauge each coordinate satisfies the wave equation: $\Box x^\mu = 0$
  - In flat space $\Box z = 0$ but $\Box r \neq 0$:
- harmonic coordinates are the extension to curved spacetime of Cartesian coordinates
- To first order in $\mathbf{h} \ll 1$ the harmonic gauge condition is

$$2\partial_\alpha h^\alpha_\nu - \partial_\nu h^\alpha_\alpha = 0$$

Einstein equations

- From Γ we construct the curvature tensor

$$R^\mu_{\alpha\nu\beta} \equiv \underbrace{\partial_\beta\Gamma^\mu_{\nu\alpha} - \partial_\nu\Gamma^\mu_{\beta\alpha}}_{} + \underbrace{\Gamma^\mu_{\beta\lambda}\Gamma^\lambda_{\nu\alpha} - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\beta\alpha}}_{}$$

$$\sim \quad \text{curl of grav. force} \qquad \Gamma \sim h \text{ so } O(h^2)$$

- Define Ricci tensor $R_{\alpha\beta} \equiv R^\mu_{\alpha\mu\beta}$
- Then Einstein field equations are

$$R_{\alpha\beta} - \tfrac{1}{2}R^\nu_\nu g_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta} \longleftarrow \text{E-p tensor}$$

Similar to $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$

Tickling spacetime

- To first order in **h**

$$R_{\alpha\beta} = \tfrac{1}{2}\left(\partial_\alpha\partial_\beta h^\lambda_\lambda - \partial_\mu(\partial_\beta h^\mu_\alpha + \partial_\alpha h^\mu_\beta) + \Box h_{\alpha\beta}\right)$$

- In the harmonic gauge $2\partial_\alpha h^\alpha_\nu - \partial_\nu h^\alpha_\alpha = 0$ this simplifies to

$$R_{\alpha\beta} = \tfrac{1}{2}\Box h_{\alpha\beta} \quad \Rightarrow \quad R^\lambda_\lambda = \tfrac{1}{2}\Box h^\lambda_\lambda$$

- The field equations are now

$$\tfrac{1}{2}\Box h_{\alpha\beta} - \tfrac{1}{4}\Box h^\lambda_\lambda \eta_{\alpha\beta} = -\frac{8\pi G}{c^4} T_{\alpha\beta}$$

- Taking the trace $\rightarrow \Box h^\lambda_\lambda = (16\pi G/c^4) T^\lambda_\lambda$

$$\Rightarrow \Box h_{\alpha\beta} = -\frac{16\pi G}{c^4}\left(T_{\alpha\beta} - \tfrac{1}{2} T^\lambda_\lambda \eta_{\alpha\beta}\right)$$

Closely analogous to $\Box A_\alpha = \mu_0 j_\alpha$

Solutions describing waves

- A plane wave propagating along x will be described by $h_{\alpha\beta}(x-ct)$
- Form of **h** must also satisfy gauge condition. A sufficiently general such form is

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & -a \end{pmatrix}$$

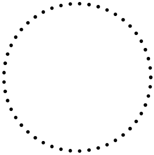
where $a(x-ct)$ and $b(x-ct)$ are arbitrary functions

- Like emag waves, gravitational waves are transverse and have 2 polarisation states:

$$a \neq 0, b = 0 \qquad \text{and} \qquad a = 0, b \neq 0$$

The wave's impact

- A ring of particle initially stationary in yz plane



- Hit by wave $\mathbf{h} = \mathrm{diag}(0, 0, a, -a)$
- Each particle's 4-velocity $\mathbf{u}$ satisfies

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = -\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$$
$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}\eta^{\mu\nu}\left(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}\right)$$

- For non-vanishing Γ need:
  - (i) two indices 2 or two 3; (ii) third index 1 or 0
- Initially $\mathbf{u} = (c, 0, 0, 0)$ and throughout 0th cpt dominates so dominant contribution to $\Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$ has α or $\beta = 0$
- Eq for u^2 dominated by $\Gamma^2_{02} = \Gamma^2_{20}$
- Conclude

$$\frac{\mathrm{d}u^2}{\mathrm{d}\tau} \simeq -\eta^{22}\left(\partial_0 h_{22} + \partial_2 h_{02} - \partial_2 h_{02}\right) c u^2 = -c\partial_0 h_{22} u^2$$

The wave's impact

- So y cpt of velocity in yz plane satisfies

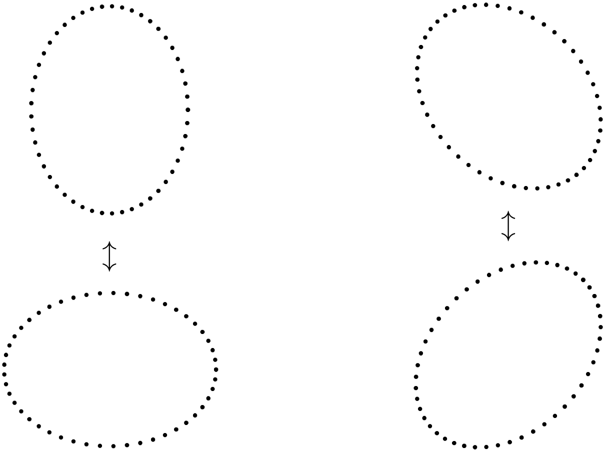
$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{\partial a}{\partial t} v_y$$

- Initially $v_y = 0$ and from above $v_y = 0$ subsequently
- It follows that y, z are unchanged by wave
- But distance between diametrically opposed points on circle *does* change:

$$D_y = 2\int_0^y \mathrm{d}y\,\sqrt{g_{22}} = 2y\sqrt{1+a}$$

- Perpendicular diameter changes differently: $D_z = 2z\sqrt{1-a}$
- So coordinates don't change but the particles do move!

$a \neq 0, b = 0$ $\qquad$ $a = 0, b \neq 0$



- Detect by comparing lengths of perpendicular diameters

Outline

Conclusions

- Gravitational waves are an inevitable consequence of Lorentz covariance
- Close parallels with emag throughout
- We find their form by expanding the potential $\mathbf{g}$ of the gravitational field around its field-free form
- It's vital to proceed in the optimum gauge - harmonic coordinates
- Perturbation to $\mathbf{g}$ satisfies wave eqn with Ep tensor as source
- The waves are transverse: squash-and-stretch
- Detect by comparing lengths of perpendicular rods